GENERAL MATHEMATICS

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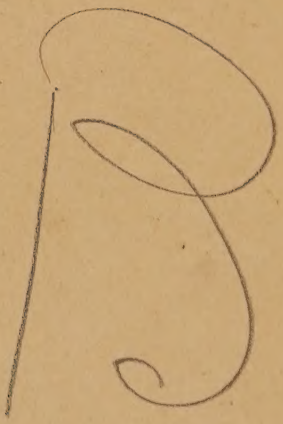

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

A COURSE IN GENERAL MATHEMATICS

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A COURSE
IN
GENERAL MATHEMATICS

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PREFACE

This book includes the elements of algebra, trigonometry, analytic geometry, and calculus. It is intended as a text for use in the freshman year of colleges and universities. It presupposes on the part of the student only the minimum entrance requirements in elementary algebra and plane geometry.

The review topics, such as the graph, simultaneous equations, and the quadratic equation, are interwoven with and related to the conic sections and other more advanced material. In the first third of the book numerical computation is made prominent. Tables, to shorten computation, have been used freely. Practical applications have been kept in mind constantly. The theory and the problem material have been selected and organized with reference to a uniform gradation of difficulty, and the order of importance to the student.

In the problems selected to illustrate the principles of analytic geometry and differential and integral calculus, the algebraic work is of the simplest character in order to direct attention to the method.

The chief features of the book are the simplicity of method and the selection and order of presentation of the subject matter. The problem material has been selected with reference to utility and interest rather than mathematical completeness.

The material presented here has been tried out in the class room and by correspondence courses during the past five years. Methods and problems which have proved unsatisfactory have been eliminated. It is therefore the hope

of the authors that this book will prove to be teachable in the hands of others and will awaken in the students an interest in the science of mathematics, which touches so many phases of human activity.

C. H. Currier
E. E. Watson

April, 1929

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A COURSE IN
GENERAL MATHEMATICS

GENERAL MATHEMATICS

CHAPTER I

FUNCTIONS AND GRAPHS

1. Introduction. In this chapter, we shall study the graphical representation of algebraic expressions and other functions. This includes the study of geometric relations by means of algebra, and the geometric illustration of the behavior of functions. These ideas are widely used, not only in mathematics, but in the various branches of science.

2. Constants. Mathematics deals with two kinds of magnitudes, constants and variables. A ***constant*** is a quantity which retains the same value throughout a given problem or discussion. *Absolute constants*, such as 5, -7, π, etc., have the same values in all problems. *Arbitrary constants*, such as the radius of a circle, may have different values in different problems.

3. Discussion of Variable and Function. If a man walks directly away from a point A at the constant rate of 4 miles per hour, his distance from A varies all the way from 0 miles to 4 miles during the first hour. His distance from A during this time has every intermediate value, and consequently an unlimited number of values in all. The relation between the distance and the time may be expressed algebraically by the equation or formula $y = 4x$, in which y represents the distance traveled in miles, and x the number of hours. For any value of x, a value of y can be found. Thus if $x = 0$, $y = 0$; if $x = 1$, $y = 4$; etc. In this equation both x and y are ***variables,*** and y is said to be a func-

tion of x, that is, the value of y depends upon the value of x. Here x is regarded as the *independent variable* and y as the *dependent variable*. In general, given two quantities as x and y, if the value of the second quantity, y, is determined when the value of the first quantity, x, is given, the second is said to be a ***function*** of the first.

TABLE

x	y
0	0
1	4
2	8
3	12
5	20

4. Notation. The symbols y, $f(x)$, or $F(x)$ are used to denote a function of x. They are read, y, or the f function of x, or the capital F function of x. Hence, if we are dealing with the expression $x^2 + 3x$, we may write:

$$y = x^2 + 3x,$$
$$f(x) = x^2 + 3x,$$

or

$$F(x) = x^2 + 3x.$$

By the expression $f(2)$ is meant the value of the f function of x when $x = 2$; thus, for the preceding example, $f(2) = 10$.

ILLUSTRATION. Given $f(x) = x^2 + x + 3$, find $f(0)$, $f(1)$, $f(2)$. Substituting 0, 1, and 2 in order for x, we obtain:

$$f(0) = 0 + 0 + 3 = 3,$$
$$f(1) = 1 + 1 + 3 = 5,$$
$$f(2) = 4 + 2 + 3 = 9.$$

EXERCISES

1. If $y = x^2 + 6x$, find y when $x = 0, 1, 2, 5, 10$.
2. If $y = x^2 - 2x + 2$, find y when $x = 0, 2, 4, 6, 8$.
3. If $f(x) = x^2 + 2$, find $f(0), f(1), f(2), f(a)$.
4. If $f(x) = x + \sqrt{x}$, find $f(0), f(1), f(4), f(16)$.
5. If $F(x) = x + 2\sqrt{x}$, find $F(0), F(1), F(25), F(100)$.
6. If $f(x) = \sqrt{x + 4}$, find $f(0), f(5), f(12), f(21)$.
7. If $F(x) = \sqrt{x^2 + 3x}$, find $F(0), F(2), F(7), F(10)$.

SUGGESTION. $F(2) = \sqrt{4+6} = \sqrt{10}$. Leave the result in this form.

8. If $y = x + 1/x$, find the values of y for each integral value of x from $x = 1$ to $x = 10$, inclusive.

9. If $f(x) = 2x + 3$, find $f(0), f(-1), f(-4), f(-10)$.

10. If $F(x) = \sqrt{x} + 1/\sqrt{x}$, find $F(1), F(4), F(10), F(100)$.

5. Methods of Representing Functional Relationships. The relation between the independent and dependent variables may be expressed in different ways.

A. By a table of corresponding values of the two variables.
B. By graphical methods. See § 6.
C. By a mathematical formula or equation.

When a sufficiently extensive table of corresponding values is given, the graph can be drawn by methods to be explained in the next paragraph. Usually the values have been obtained by computation, by careful measurement, or by actual counting.

6. Representation of Statistics. Simple statistical material can be presented in tabular form or in graphical form, as shown in the following example, which presents the changes in the population of a certain city during the period 1870–1920.

YEAR	POPULATION
1870	5 000
1880	10 000
1890	20 000
1900	22 000
1910	35 000
1920	40 000

TABULAR METHOD

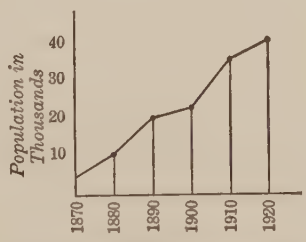


FIG. 1. Graphic Representation.

Graphically the population at each of the given dates is represented by a line-segment. For convenience of com-

parison, these segments are placed vertically. The use of cross-section paper facilitates the plotting and makes the diagram easier to read. The scale should be clearly shown.

EXERCISES

1. According to the records of the Weather Bureau, the following was the temperature record for the first fifteen days of June, 1920, in Providence.

DATE. June 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.
TEMPERATURE.
Highest. 86, 85, 82, 67, 52, 63, 66, 62, 67, 65, 91, 80, 62, 81, 90.
Lowest. 55, 62, 63, 51, 49, 48, 51, 51, 51, 50, 56, 62, 55, 53, 67.
Mean. 70, 74, 72, 59, 50, 56, 58, 56, 59, 58, 74, 71, 58, 67, 78.

Plot the curve for (*a*) highest daily temperature,
(*b*) lowest daily temperature,
(*c*) mean daily temperature.

2. For the same month and place the percentage of sunshine was recorded as follows:

DATE. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.
PER CENT OF
SUNSHINE. 84, 100, 53, 24, 0, 3, 46, 12, 100, 76, 57, 83, 0, 83, 68.

Plot the curve.

3. According to the American Experience Table of Mortality, used by most American life insurance companies, out of 100 000 people living at age 10 the following table gives the number living at the various ages mentioned:

AGE	NUMBER LIVING	AGE	NUMBER LIVING	AGE	NUMBER LIVING
10	100 000	40	78 106	70	38 569
20	92 637	50	69 804	80	14 474
30	85 441	60	57 917	90	847
				95	3

Plot the corresponding curve. Estimate from it the number living at age 42. At what age are 50 000 still living?

4. The pressure of water per square inch at different depths is given by the following table:

DEPTH IN FEET	PRESSURE IN LB.	DEPTH IN FEET	PRESSURE IN LB.
6	2.60	45	19.49
10	4.33	50	21.65
20	8.66	60	25.99
30	12.99	70	30.32
40	17.32	80	34.65

Show graphically how pressure varies with depth. Estimate the pressure at a depth of 28 ft. What is the depth corresponding to a pressure of 20 lb.?

5. The population of the United States at each census from 1790 to 1920 is given in the following table, correct to the nearest hundred-thousand:

1790	3 900 000	1860	31 400 000
1800	5 300 000	1870	38 600 000
1810	7 200 000	1880	50 200 000
1820	9 600 000	1890	62 900 000
1830	12 900 000	1900	76 000 000
1840	17 100 000	1910	93 000 000
1850	23 200 000	1920	105 700 000

Plot the population curve. Estimate from the graph the population in 1865.

6. The amount of $100 placed at compound interest for 10 years at various rates is shown by the following table:

Rate Per Cent	Amount of $100 after 10 Years	Rate Per Cent	Amount of $100 after 10 Years
2	$121.90	6	$179.08
3	134.39	8	215.89
4	148.02	10	259.37
5	162.89	20	619.17

Plot the amount as a function of the rate per cent.

7. Advantages of Graphical Representation. A consideration of the problems already solved will show that the graphical method exhibits the actual variation of the function more clearly than do tables of values. For this reason, in scientific research the data are usually graphed and the graph carefully studied to note general tendencies as well as peculiarities at certain points. In this way the underlying scientific law frequently can be discovered.

8. Directed Lines. In laying off distances from a point on a straight line, it is often useful to have some way of determining which direction is to be taken. This is done by prefixing to each number used a label, or sign; $(+)$ if the distance is taken in one direction, and $(-)$ if in the opposite

direction. If north is taken as positive (+), south is negative (−); if up is taken as positive (+), down is negative (−); if the direction A to C is positive, C to A is negative.

9. Relations Between Segments of Directed Lines. In plane geometry we spoke of a line-segment as AB or BA at will. In a directed line, however, if AB represents a positive number, BA is negative, that is, we have the relation $AB = -BA$.

Moreover, for any three points A, B, C on a directed line, no matter what their relative positions, we have $AB + BC$

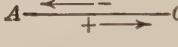


Fig. 2.

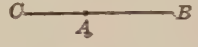


Fig. 3.

$= AC$. To show that this is true, we must consider the six possible orders in which A, B, C may be written on a straight line, namely: ABC, ACB, BAC, BCA, CBA, and CAB. In the last case, we have (see Fig. 3):

$$AB + BC = AB + BA + AC = AB - AB + AC = AC.$$

10. Axes. Two lines perpendicular to each other and lying in the same plane may be taken as *lines of reference* or ***axes.*** These lines are usually designated by XX' and YY', and are called the x-axis, and the y-axis, respectively. The point of intersection, O, is called the ***origin.*** It serves as a convenient starting point from which to measure distances.

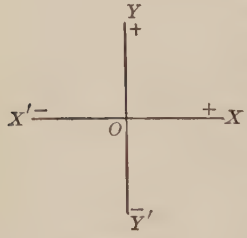


Fig. 4.

11. Quadrants. The axes produced divide the plane into four parts, called ***quadrants.*** The part of the plane which lies to the right of the y-axis and above the x-axis is

called the *first quadrant*, and is designated by *I*, Fig. 5. The other quadrants are numbered II, III, IV, as shown in the figure.

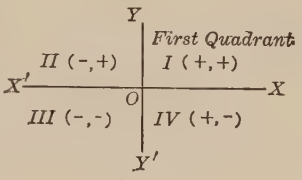


Fig. 5.

12. Rectangular Coordinates of a Point. From any given point, *P*, in the plane, draw a line parallel to the *y*-axis, intersecting the *x*-axis in some point *N*. Then *ON* is the *x*-distance or *abscissa* of the point *P*, and *NP* is the *y*-distance, or *ordinate*, Fig. 6. Taken together, the abscissa and ordinate are called the ***coordinates.*** They are written, for brevity, within parentheses and separated by a comma, the abscissa always being written first, as (x, y).

Paper ruled in squares, called coordinate paper, is convenient when working with rectangular coordinates. Locat-

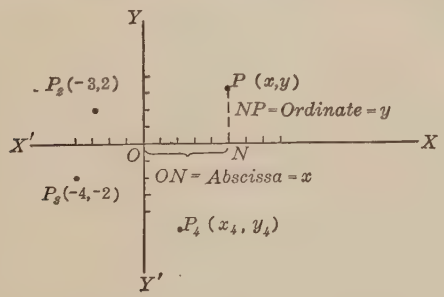


Fig. 6.

ing points on a drawing by means of their coordinates is called *plotting* the points.

13. Signs of the Coordinates. Abscissas are considered positive if measured to the right of the *y*-axis, and negative if measured to the left. Ordinates are considered positive

if measured upward from the x-axis, and negative if measured downward from the x-axis.

The signs of the coordinates in the different quadrants are shown in the following table:

Quadrants	I	II	III	IV
Abscissa	+	−	−	+
Ordinate	+	+	−	−

14. Historical Note. The use of coordinates is very old. They were used in the discussion of longitude and latitude by Hipparchus about 140 B. C., and in the methods of surveying used by Heron of Alexandria about 50 A. D. In 1637, Descartes, a noted French mathematician and philosopher, published the system of representing points by number pairs (x, y) and lines by equations based on coordinates. Since then, algebra and geometry have been more closely united through the graph and graphical representation.

EXERCISES

1. Draw a pair of coordinate axes and locate the following points: (5, 2), (6, 4), (−4, 3), (5, −2), (8, −3), (−6, 4), (7, −5), (6, 0), (−7, 0), (0, 5), (0, −5), (0, 0).

2. Plot the points, (4, 0), (−2, 8), (5, 4), (6, −7), (−5, −8), (−3, −6).

3. A line which has one end at the point (5, −2) is bisected at the origin. What are the coordinates of the other end?

4. What are the coordinates of a point symmetrical to the point (−3, 4) with respect to: (*a*) the y-axis?, (*b*) the x-axis?, (*c*) the origin?

5. Where are all the points whose abscissas are 5? Whose ordinates are 2? Whose abscissas are −2? Whose ordinates are −3? Whose abscissas are 0? Whose ordinates are 0?

6. In what quadrant is a point: (*a*) if its abscissa is positive and its ordinate is negative?, (*b*) if both coordinates are positive?, (*c*) if both coordinates are negative?

7. Describe the position of a line on which the following points are located:

(*a*) (1, 1), (−3, −3), (4, 4), (−5, −5);
(*b*) (2, −2), (−3, 3), (5, −5), (−4, 4).

8. Plot the points (4, 3), (5, 0), (−4, −3), (4, −3), (0, 5). Show that they are equally distant from the origin, and therefore lie on the circumference of a circle whose center is at the origin. Give the coordinates of other points on the same circle.

9. A square whose side is 5 units is placed with one vertex at the origin, and an adjacent side on the x-axis. Find the coordinates of the other vertices. (Four solutions.)

10. An equilateral triangle whose side is 6 units is placed with one vertex at the origin and an adjacent side on the x-axis. Find the coordinates of the other vertices. How many solutions are there?

11. A square whose side is 10 units long has its center at the origin. What are the coordinates of its vertices if (*a*) the sides are parallel to the axes?, (*b*) the diagonals coincide with the axes?

12. One end of a line bisected at the origin is the point (−2, −3). What are the coordinates of the other end?

13. Two vertices of an equilateral triangle are at (2, 4) and (4, 4). What are the coordinates of the third vertex?

14. Two diagonally opposite vertices of a square are at (3, 0) and (−3, 0). What are the coordinates of the other vertices?

15. Graphical Representation of Functions. To construct the graph of a function, values of the variable (x) are taken as abscissas of points, and the corresponding values of the function (y) as ordinates. A smooth curve which contains these points in order is called the graph of the function.

16. Linear Equations and their Graphs. Let it be required to construct the graph of the equation $y = 2x - 3$.

SOLUTION. Let x take such values as 0, 1, 2, 3, 4, etc., and find the corresponding values of y.

Thus if	$x =$	0	1	2	3	4	6
Then	$y =$	−3	−1	1	3	5	9

The line joining these points is the graph of $y = 2x - 3$. See Fig. 7. This process is called plotting the graph of the equation, or simply plotting the equation. In general, we shall see that every equation of the first degree may be represented by a straight line. For this reason first degree equations are commonly called linear equations.

17. Two Points Determine the Graph of a Straight Line. Since a straight line is fixed by any two points on it, to find the graph of an equation of the first degree, we need to find

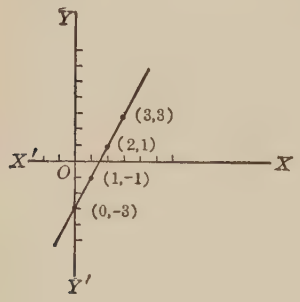


Fig. 7.

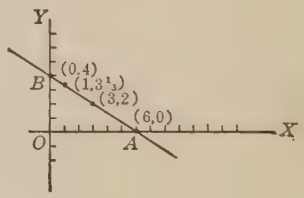


Fig. 8.

only two points on the graph and draw a straight line through them.

Example. Plot the graph of $2x + 3y = 12$. For $x = 1$, $y = 3\frac{1}{3}$; for $x = 3$, $y = 2$. Locate the points $(1, 3\frac{1}{3})$ and $(3, 2)$ and draw a straight line through them. See Fig. 8.

18. Intercepts. The distances OA, and OB, cut off on the axes by a curve or straight line are called ***intercepts,*** Fig. 8. Since $x = 0$ for every point on the y-axis, to find the y-intercept put $x = 0$ and solve for y. Similarly to find the x-intercept put $y = 0$ and solve for x.

In the preceding example, $2x + 3y = 12$, if $x = 0$, $y = 4$; if $y = 0$, $x = 6$. Hence the x-intercept is 6; and the y-intercept 4.

EXERCISES

1. Plot the graph of each of the following equations, using several values of x and y, at least one of which should be negative.

(a) $y = x - 2$.	(d) $y = \frac{1}{2}x + 6$.	(g) $4x + 5y = 20$.
(b) $y = 2x + 4$.	(e) $y = 3x - 12$.	(h) $3x + 4y = 12$.
(c) $3x + 5y = 15$.	(f) $4x - 5y = 20$.	(i) $2x - 7y = 14$.

2. Which of the points (1, 3), (3, 7), (0, 1), (−1, −2), (5, 11), $(-\frac{1}{2}, 0)$, (8, 17), (9, 20), (−3, −5), lie on the graph of $y = 2x + 1$?

3. The amount of \$100 at simple interest after t years, interest at 8%, is $A = 100 + 8t$. Plot the graph and find the amount at the end of 2 years, 10 years.

4. The charge for printing n hundred circulars of a certain kind is $p = 3n + 2$ dollars. Represent the equation graphically.

5. Plot the graph of each of the following equations. Time yourself.

(a) $x + 3y = 6$.	(g) $3x - 5y = 15$.
(b) $4x + y = 8$.	(h) $x - 2y = 8$.
(c) $2x + 3y = 18$.	(i) $2x + 3y = -12$.
(d) $x + 5y = 10$.	(j) $3x + 7y = -21$.
(e) $2x - 3y = 12$.	(k) $4x - 7y = -28$.
(f) $4x - 2y = 20$.	(l) $2x - 5y = -10$.

6. Find the intercepts for each of the following equations and plot the graph.

(a) $x + y/2 = 4$.	(g) $x/5 - y/2 = 10$.
(b) $2x + y/3 = 6$.	(h) $x/3 + y/4 = 1$.
(c) $2x + y/2 = 4$.	(i) $x/5 - y/3 = 1$.
(d) $x - y/3 = 9$.	(j) $x/3 - y/7 = 1$.
(e) $2x - y = 6$.	(k) $x = 6 - 2y$.
(f) $x/3 - 2y = 4$.	(l) $3x = 5y - 15$.

7. A pressure of 1 pound per square inch is equivalent to a column of 2.034 inches of mercury, or to one of 2.304 feet of water. Draw a graph showing the relation between pressure expressed in feet of water and pressure expressed in inches of mercury.

8. The pressure of a liquid is proportional to the depth. If the pressure per square inch on the suit of a diver is 5.2 lb. for a depth of 12 ft., how deep can he go safely if 78 lb. per square inch is an allowable pressure? Construct the graph.

19. The Equation of the Circle. The equation of the circle in terms of its radius and the coordinates of its center is easily determined. In Fig. 9 let $O'(a, b)$ represent the center of the circle, r its radius, and $P(x, y)$ any point on the circumference of the circle. Join O' and P. Through P draw PN perpendicular to the x-axis, and through O' draw $O'C$ parallel to the x-axis. Then in the right triangle $O'CP$,

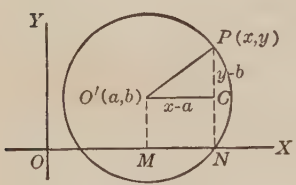


FIG. 9.

$$(1) \qquad (O'C)^2 + (CP)^2 = r^2.$$

But $O'C = ON - OM = x - a$, and $CP = NP - NC = y - b$. Therefore

$$(2) \qquad (x - a)^2 + (y - b)^2 = r^2.$$

This is the required equation, for it expresses the relation between (a, b), the coordinates of the center, and (x, y), the coordinates of any point on the circumference.

It may be written in the form

$$(3) \qquad x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0,$$

or

$$(4) \qquad x^2 + y^2 + Dx + Ey + F = 0,$$

where

$$(5) \qquad D = -2a, \quad E = -2b, \quad \text{and} \quad F = a^2 + b^2 - r^2.$$

EXAMPLE. The equation of a circle whose center is at $(3, -2)$ and whose radius is 5, is readily seen to be

$$(x - 3)^2 + (y + 2)^2 = 25.$$

If the center of the circle is the origin, then $a = 0$, $b = 0$ and the equation (see Fig. 10) is

$$x^2 + y^2 = r^2.$$

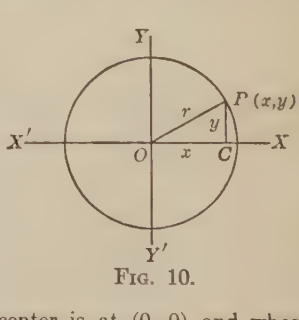


FIG. 10.

Thus the equation of a circle whose center is at (0, 0) and whose radius is 7, is $x^2 + y^2 = 49$.

20. To Construct a Circle from its Equation. Reduce the equation of the circle to the form $(x - a)^2 + (y - b)^2 = r^2$. Locate the point (a, b). With this point as a center and a radius equal to r describe a circle. This will be the required circle.

EXAMPLE. Draw the circle whose equation is

$$2x^2 + 2y^2 - 12x - 16y + 32 = 0.$$

SOLUTION. Dividing the equation by 2, we find

$$x^2 - 6x + y^2 - 8y + 16 = 0.$$

Completing the squares in x and y, we have

$$(x^2 - 6x + 9) + (y^2 - 8y + 16) = 9,$$

or

$$(x - 3)^2 + (y - 4)^2 = 3^2.$$

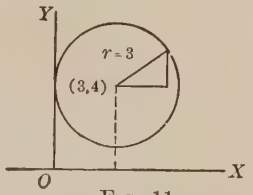


FIG. 11.

The required circle has its center at (3, 4), and has a radius equal to 3, (Fig. 11).

EXERCISES

1. Draw the graph of each of the following equations. Find the radius and use a compass to draw the graph.

(*a*) $x^2 + y^2 = 16$. (*c*) $x^2 + y^2 = 64$. (*e*) $x^2 + y^2 = 20\frac{1}{4}$.
(*b*) $x^2 + y^2 = 49$. (*d*) $x^2 + y^2 = (4.2)^2$. (*f*) $x^2 = 30\frac{1}{4} - y^2$.

2. Find the equation of a circle having its center and radius as follows.

(*a*) Center (3, 4), radius 7. (*e*) Center (6, 0), radius 6.
(*b*) Center (2, 3), radius 5. (*f*) Center (0, 5), radius 8.
(*c*) Center (3, −5), radius 4. (*g*) Center (−2, −5), radius 10.
(*d*) Center (−2, 6), radius 8. (*h*) Center (−4, 6), radius 5.

3. Locate the center, and find the radius of each of the following circles. Draw the graph.

(*a*) $(x - 13)^2 + (y + 8)^2 = 4$. (*e*) $x^2 - 10x + y^2 = 0$.
(*b*) $(x - 8)^2 + (y - 7)^2 = 9$. (*f*) $x^2 + y^2 - 8y = 0$.
(*c*) $(x - 6)^2 + (y - 6)^2 = 36$. (*g*) $x^2 + y^2 - 4x - 6y = 12$.
(*d*) $(x + 6)^2 + (y - 6)^2 = 36$. (*h*) $x^2 + y^2 + 6x - 8y = 24$.

4. Write the equations of the following circles:

(a) tangent to the x-axis at the origin, radius 5,
(b) tangent to the y-axis at the origin, radius 7,
(c) center at (6, 6) and passes through the origin,
(d) tangent to both axes, center at (4, 4).

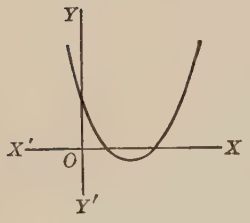


Fig. 12.

21. Quadratic Equations. The graph of every quadratic function in one unknown belongs to the class of curves known as parabolas. The parabola resembles in form an oval, open at one end. Its graph may (a) cross the x-axis at two points, (b) be tangent to the axis, (c) lie wholly above or below the axis.

Example. Plot the graph of $y = x^2 - 4x + 3$. If $x = 0$, $y = 3$; if $x = 1$, $y = 0$; if $x = 2$, $y = -1$; if $x = 3$, $y = 0$; if $x = 4$, $y = 3$, etc. A smooth curve which contains these points is called the graph of the function.

x	y
0	3
1	0
2	−1
3	0
4	3

EXERCISES

1. On the same axes plot the graphs of $y^2 = x$, $y^2 = -x$, $x^2 = y$, $x^2 = -y$.

2. From the graph of $y = x^2$, find approximately the square roots of 18, 23, 28, 32, 40, 50, 56, 60, 24, 28, 72, 65, 20, 10.

3. Plot the graph of:

(a) $y = x^2 - 4$. (b) $y = x^2 - 9$. (c) $y = x^2 + 1$. (d) $y = x^2 - 2x$. (e) $y = x^2 - 4x$. (f) $y = x^2 - 5x$. (g) $y = x^2 - 3x$. (h) $y = x^2 - 5x + 6$. (i) $y = x^2 - 7x + 10$.

22. Systems of Linear Equations in Two Unknowns. The equation $x + 2y = 7$ has an indefinitely large number of solutions. For every value of one of the unknowns, a value of the other can be found that satisfies the equation. In

general if the graphs of two linear equations are drawn, the point of intersection will have as its coordinates a pair of values (x, y), which satisfy both equations. This pair of values is the common solution. In Fig. 13 the point of intersection of the graphs is $x = 3$, and $y = 2$. These values of x and y are the values that would be obtained by using one of the methods common in elementary algebra, namely:

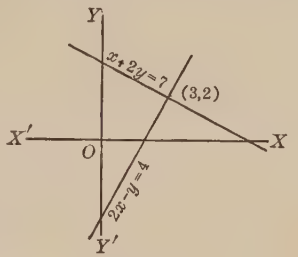


Fig. 13.

(*a*) Elimination by addition and subtraction,
(*b*) Elimination by substitution.

The student should make sure that he can use methods (*a*) and (*b*) before proceeding further.

To determine graphically the common solution of two equations having two variables:

1. Draw, with reference to a set of axes, the graph of each equation.
2. Find the coordinates of the point common to the two graphs. This is the common solution.
3. Test the common solution by substituting in each equation. Substituting 3 for x, and 2 for y, in the above equations, we obtain, $3 + 4 = 7$, and $6 - 2 = 4$. Hence both equations are satisfied.

EXERCISES

Determine graphically, and by one other method, the common solution of each of the following nine pairs of linear equations. Check the results.

1. $\begin{cases} x + 2y = 12, \\ x - y = -3. \end{cases}$
2. $\begin{cases} 5x + 2y = 11, \\ 6x - 2y = 0. \end{cases}$
3. $\begin{cases} x + 7y = 6, \\ 2x - 3y = 12. \end{cases}$
4. $\begin{cases} 2x + 5y = 20, \\ 2x - y = -4. \end{cases}$
5. $\begin{cases} x + 2y = 0, \\ x + y = 3. \end{cases}$
6. $\begin{cases} x - y = -1, \\ 2x + 3y = 18. \end{cases}$

7. $\begin{cases} x + 3y = -13, \\ x - 2y = 12. \end{cases}$ 8. $\begin{cases} 3x + 2y = 8, \\ 5x - y = -4. \end{cases}$ 9. $\begin{cases} 5x - 4y = -1, \\ x + y = 7. \end{cases}$

Solve each of the following pairs of simultaneous equations. Check the results.

10. $\begin{cases} 5x + 6y = 31, \\ x + 7y = 12. \end{cases}$

11. $\begin{cases} 2x - y - 7 = 0, \\ 5x - 7y = 13. \end{cases}$

12. $\begin{cases} \dfrac{11x - 5y}{11} = \dfrac{3x + y}{16}, \\ 8x - 5y = 1. \end{cases}$

13. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{5} = 9, \\ \dfrac{x}{4} + \dfrac{y}{5} = 7. \end{cases}$

14. $\begin{cases} 3x - 2y - 5 = 0, \\ x = 6y - 9. \end{cases}$

15. $\begin{cases} x + y = 3, \\ 2x + 3y = 1. \end{cases}$

16. $\begin{cases} \dfrac{-x}{5} + \dfrac{y}{2} = 4, \\ \dfrac{x}{3} + \dfrac{y}{7} = 7. \end{cases}$

17. $\begin{cases} \dfrac{x - 1}{3} = \dfrac{y - 2}{-8}, \\ 2x + y = 2. \end{cases}$

18. $\begin{cases} ax + by = c, \\ dx + ey = f. \end{cases}$

23. One Equation Linear and the other Quadratic. Let it be required to solve the simultaneous equations

$$x - y = 1, \tag{6}$$

$$x^2 + y^2 = 25. \tag{7}$$

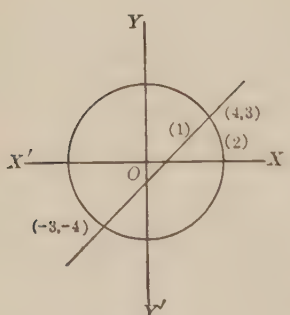


FIG. 14.

(*a*) GRAPHICAL METHOD. Draw the graph of each equation, then observe the x and the y of each of the points where the two graphs intersect. The particular pair (or pairs) of values (x, y) thus obtained constitute the solution. The graphs of equations (6) and (7) intersect at two points, and therefore the equations have two solutions, namely: $x = 4$, $y = 3$; and $x = -3$, $y = -4$.

(*b*) ELIMINATION BY SUBSTITUTION. Solving the equation (6) for y, $y = x - 1$. Substituting this value of y for y in equation (7) gives $x^2 + (x - 1)^2 = 25$. Expanding and collecting, $2x^2 - 2x - 24 = 0$. Dividing by 2 and solving for x, $x = 4$ or -3. From equation (6) if $x = 4$, $y = 3$; if $x = -3$, $y = -4$. Hence the required solutions are (4, 3) and (−3, −4).

24. Both Equations Quadratic. Methods similar to the ones in § 23 often may be employed. They will lead to no real solution (roots imaginary), one real solution, two real solutions, three real solutions or four real solutions, depending upon the equations used.

EXAMPLE. Solve the system

$$\begin{cases} x^2 + y^2 = 32, \\ y^2 = 4x. \end{cases}$$

SOLUTION. Eliminating y^2 by substitution, we have

$$x^2 + 4x - 32 = 0.$$

Factoring, we find

$$x - 4 = 0, \quad \text{or} \quad x + 8 = 0.$$

Whence

$$x = 4, \quad \text{or} \quad -8.$$

Substituting in the second of the given equations, we find

$$y = \pm 4, \quad \text{or} \quad \pm 4\sqrt{-2}.$$

It will be seen that $x = -8$ corresponds to imaginary values of y; hence the real solutions are $(x = 4, \quad y = 4)$ and $(x = 4, \quad y = -4)$. The graphical method (see Fig. 15) gives the real solutions only.

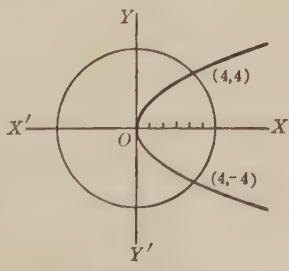


FIG 15.

While no general method is given for solving two simultaneous quadratic equations, it is advantageous to examine the equations with a view to

making such combinations as will yield one or more new systems each of which can be solved by methods already familiar. All solutions obtained in this way should be checked by substituting in the equations, in order to avoid false combinations of the x and y values thus obtained.

EXERCISES

Solve each of the following systems (a) graphically, (b) by the method of substitution. In each case the resulting quadratic equation may be solved by factoring. Verify the results.

1. $\begin{cases} x^2 + y^2 = 40, \\ x + y = 8. \end{cases}$
2. $\begin{cases} x^2 + y^2 = 29, \\ x - y = 3. \end{cases}$
3. $\begin{cases} x^2 + y^2 = 72, \\ y^2 = 6x. \end{cases}$
4. $\begin{cases} y^2 = 4x, \\ x = 4. \end{cases}$
5. $\begin{cases} y^2 = 8x, \\ y = 2x. \end{cases}$
6. $\begin{cases} x^2 + y^2 = 10, \\ x^2 = 9y. \end{cases}$
7. $\begin{cases} x^2 = 6y, \\ y^2 = 6x. \end{cases}$
8. $\begin{cases} x^2 = -6y, \\ y = -6x. \end{cases}$
9. $\begin{cases} 3x^2 + y^2 = 28, \\ x - y = -2. \end{cases}$

Determine graphically, and by one other method, the common solutions of the following pairs of equations. Check the results.

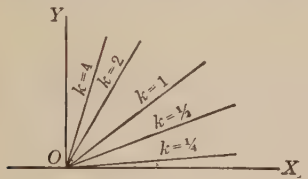
10. $\begin{cases} x^2 + y^2 = 25, \\ x + y = 7. \end{cases}$
11. $\begin{cases} y^2 = 5x, \\ x^2 = 5y. \end{cases}$
12. $\begin{cases} x^2 + y^2 = 32, \\ y^2 = 4x. \end{cases}$
13. $\begin{cases} x^2 = 4y, \\ y = 4. \end{cases}$
14. $\begin{cases} x^2 + y^2 = 50, \\ x^2 = 5y. \end{cases}$
15. $\begin{cases} x^2 + y^2 = 98, \\ y^2 = 7x. \end{cases}$

25. Variation of Functions. It is common in scientific work to find that a change in one variable produces a change in another. Thus the growth of a plant depends upon the soil, light, temperature, and humidity. The volume of a gas contained in an elastic bag depends upon the pressure and the temperature. The pressure of the wind per square unit of area depends upon the square of the velocity of the wind.

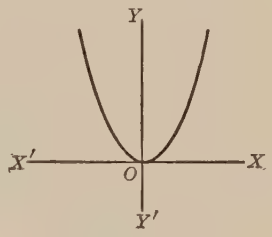
To use mathematics effectively in such problems, one must be able to translate these relations into equations.

26. Direct Variation. The simplest relation that can exist between two variables is expressed by the equation

$y = kx$, in which each variable is said to vary directly as the other, that is, the doubling of one doubles the other and so on. Relations of this kind occur very frequently in ordinary life. Thus if k represents the price of one pound of sugar, x the number of pounds bought, and y the cost of x pounds then $y = kx$. If k is the rent of a house for one month, the rent for x months is $y = kx$.



Geometric Representation of $y = kx$.
FIG. 16.



Graph of $x^2 = 4y$.
FIG. 17.

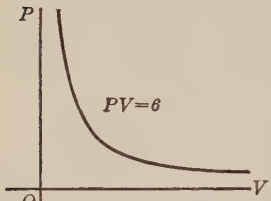
27. Graphical Representation. If y varies as x, then $y = kx$. The graph is a straight line through the origin. Figure 16 shows the graph of $y = kx$, where $k = \frac{1}{4}, \frac{1}{2}, 1, 2,$ and 4. The constant k may be determined by some condition in the problem.

EXAMPLE. Given that y varies as x^2, and that $y = 4$, when $x = 4$, find the equation connecting x and y.

SOLUTION. The relation is expressed by the equation $y = kx^2$. Since $y = 4$, when $x = 4$, we obtain by substitution $4 = 16k$, whence $k = \frac{1}{4}$, and the required equation is $y = x^2/4$, or $x^2 = 4y$. The graph is the curve called a *parabola*, Fig. 17.

28. Inverse Variation. When the product of two variables is constant, as $xy = k$ (that is, $y = k/x$, or $x = k/y$), each variable is said to vary inversely as (or as the reciprocal of) the other. The graph of the relation is a curve called a rectangular hyperbola. As in the straight line, one point is sufficient to determine k and fix the curve.

Boyle's law states that the volume V of a given quantity of gas at a constant temperature varies inversely as the pressure, that is, $PV = k$, or $V = k/P$. The intensity (I) of light varies inversely as the square of the distance of the source, i.e., $I = k/d^2$. If $z = kx/y$, z is said to vary directly as x and inversely as y.



Graph of $PV = 6$.

FIG. 18.

EXAMPLE. Given that the volume V of a given quantity of gas varies inversely as the pressure P, and that $V = 6$ cubic in., when $P = 1$ lb., find the equation and draw the graph.

SOLUTION. From the conditions given, $PV = k$, also $P = 1$, when $V = 6$. Hence $6 \times 1 = k$, or $k = 6$, and the equation is $PV = 6$. The graph for positive values of P and V is shown in Fig. 18.

29. Joint Variation. One quantity is said to vary jointly as two others when it varies directly as their product. Thus the area of a rectangle varies jointly as its length and width, i.e., $A = klw$.

If $z = kxy$, then z is said to vary jointly as x and y.

The formula may be readily extended to include any number of variables.

EXERCISES

Express the following statements in the form of equations:

1. The amount of gasoline used by an automobile varies directly as the number of miles traveled.

2. At a given time, the height of an object varies as the length of its shadow.

3. The area of a sphere varies directly as the square of its radius.

4. The arc of a given circle varies as its central angle.

5. The time required for a pendulum to make one vibration varies as the square root of the length. If a pendulum 100 cm. long vibrates seconds, what is the time of vibration of a pendulum 64 cm. long? 36 cm. long? 144 cm. long?

6. The area that can be covered by an ounce of gold leaf varies inversely as the thickness of the gold leaf.

7. The number of vibrations a pendulum makes in one second varies inversely as the square root of the length.

8. The volume of a cone having a circular base varies jointly as the altitude and the square of the radius of the base.

9. The resistance of a wire to an electric current varies directly as its length and inversely as the area of its cross section.

10. The weight of a body varies directly as its mass and inversely as the square of its distance from the center of the attracting body.

Solve the following exercises:

11. The area of a circle varies as the square of its diameter. How is the area affected when the diameter is doubled? Tripled? Halved?

12. If a city is receiving its water supply by means of a main from a reservoir, show that the supply can be approximately doubled by increasing the diameter of the main 40%. What would be the effect of increasing the diameter by 12%?

13. Hooke's law states that the amount of stretching in an elastic string varies as the tension. If a 15 lb. weight will stretch a string 6 inches, find the tension when the stretching is 1 inch, 10 inches. How could you graduate (divide) a scale for the spring-balance in the above problem?

14. The elongation of a spring-balance varies as the weight applied. If $E = 6$ when $W = 30$, find the relation between E and W.

15. In enlarging a photograph, the original negative is projected on a sensitized plate, just as a lantern slide is projected on a screen. If the negative is 10 inches from the point source of light, where should the sensitized plate be put in order that its *area* may be 4 times as large? 9 times as large? 100 times as large?

16. The maximum range (R) of a projectile varies as the square of the velocity (V) with which it starts. If the velocity is 400 ft. per second, the range is 5000 ft. (neglecting atmospheric resistance). Find the range when the velocity is 800 ft. per second. 2400 ft. per second.

17. The distance a body falls, starting from rest, varies as the square of the time. If a body falls 16 ft. during the first second, how far will it fall in 2 sec.? 5 sec.? 10 sec.?

18. The volume of a sphere varies as the cube of its diameter. How

many shot $\frac{1}{8}$ of an inch in diameter can be made by melting a lead sphere 2 inches in diameter? From one 4 inches in diameter?

19. The weight of an iron ball varies as the cube of its diameter. If 1000 lb. of iron will make a ball 20 in. in diameter, what will be the diameter of an iron ball weighing 125 lb.?

20. The pressure P (in pounds) of an air current upon a flat surface one foot square and perpendicular to the current is given by the formula:

$P = 0.003v^2$, when v is expressed in miles per hour,

or

$P = 0.0014v^2$, when v is expressed in feet per second.

Find the pressure for wind velocities of (*a*) 10, 20, 30, 40, 50, 100 miles per hour, (*b*) 40, 60, 100 feet per second.

21. The intensity of light varies inversely as the square of the distance from the source of light. If an object is now 40 ft. from the light, how far from the light must it be to receive 4 times as much light? $\frac{1}{4}$ as much light?

22. If the amount of light and heat received per square unit of surface at the distance of the earth from the sun be taken as 1, how much light per square unit would each of the other planets receive, the distances being as follows in millions of miles?

Mercury.....36	Mars 142	Uranus.....1782
Venus.....67	Jupiter.....483	Neptune.....2792
Earth.....93	Saturn.....886	

23. If two pulleys are connected by a belt, prove that the number of revolutions they make per minute are to each other inversely as their diameters.

24. An electric current varies inversely as the resistance of the circuit that is, $c = k/r$. If $c = 2$ (amperes) when $r = 5$ (ohms), find c when $r = 1;\ 4;\ 10;\ \frac{1}{2};\ \frac{1}{4}$. Draw a graph showing this relation when $k = 2$.

25. The electrical resistance of a wire varies inversely as the square of the diameter x. If $R = 12$ when $x = 3$, find the formula connecting R and x.

26. An electric current varies inversely as the resistance. If $C = 4$ when $R = 28$, find the equation connecting C and R. Find C when $R = 6$, $R = 14$, $R = 60$.

27. A force acts perpendicular to a lever at its outer end. If the turning moment is constant and equal to 12, plot a graph showing the relation between the force, x lb., and the length of the lever arm y.

28. The pressure of wind on a sail varies jointly as the area of the sail and the square of the wind's velocity. When the velocity of the wind is 18 miles per hour, the pressure on a square foot is 1 lb. What is the velocity of the wind when the pressure is 4 lb.? 25 lb.? 100 lb.?

29. The safe load of a horizontal beam supported at both ends varies as the breadth and the square of the depth, and inversely as the length between the supports. If a 2×6 inch white pine joist, 10 ft. long, safely holds 800 lb., what is a safe load for a 4×8 joist of the same material, 12 ft. long?

30. If a body is attached to a string and then swung round and round in a circle, the outward pull P varies directly as the mass and the square of the velocity, and inversely as the radius, $P = kmv^2/r$. How is P affected by doubling the mass m?

CHAPTER II

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

30. Use of Graphical Methods and Natural Functions. A triangle is composed of six parts, three sides and three angles, and may be solved, that is, the values of all six parts can be found, if three parts are given, at least one of which is a side. In a right triangle, one part, the right angle, is always given; hence a right triangle can be solved if (*a*) two sides, or (*b*) one side and an acute angle are given.

There are in common use two methods of solving triangles, namely:

(*a*) The graphical method (scale drawings).

(*b*) The trigonometric method, in which the unknown parts are obtained by computation.

One of the important applications of trigonometry is in solving triangles.

In connection with graphical methods, it is assumed that the student is fairly familiar with the principle of scale-drawings. From such a drawing one can often obtain dimensions or measurements of distances which it is difficult, or even impossible, to measure directly. To the student of mathematics or science, scale drawings are useful: (*a*) in estimating results, (*b*) as an aid in visualizing the conditions of the problem, (*c*) as a check on the results obtained by computation.

31. Graphical Solutions. Ordinary surveying problems can be solved approximately by drawing the figure to scale and reading off the required distance or angle. With good instruments the error need not be large, and the method can

be used to check roughly the results obtained by more refined methods.

EXAMPLE. At a distance of 100 feet from the foot of a tree, the angle of elevation of the top of the tree is 28°; find the height of the tree.

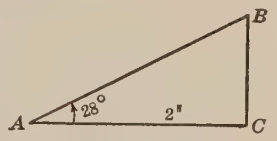


FIG. 19.

SOLUTION. Let the scale be 1″ = 50 ft. Then AC, the distance from the foot of the tree, will be represented by a line 2″ long. Angle $CAB = 28°$. The perpendicular BC which represents the height of the tree is found by measurement to be 1.06 in., nearly. Hence the height of the tree is $50 \times 1.06 = 53$ ft.

The graphical method of solving right triangles, although exceedingly simple and useful, is not sufficiently accurate for all purposes. There exists another method of solving triangles by which the unknown parts of a triangle are computed from the numerical values of the given parts. This enables one to obtain results correct to any desired degree of accuracy.

32. Trigonometry. Trigonometry is the basis of surveying and of many calculations in engineering, physics, astronomy, and other sciences. The word *trigonometry* is derived from Greek words which together mean "measure of a triangle." Its principal object is to find the remaining parts of a triangle when three parts are given, one of which must be a side. The student has seen that geometry shows how, from three such parts, to construct the triangle, and also that these parts are mutually dependent. Trigonometry will enable him to compute the unknown parts from the numerical values of the given parts, and also to find the exact nature of this dependence.

33. Definitions of the Six Ratios. We usually denote the angles of a right triangle by the capital letters A, B, C (C being the right angle), and the sides opposite these angles

by the small letters a, b, c, respectively. The six ratios among the sides a, b, c are called the ***trigonometric functions*** of the acute angle A, and are defined as follows:

(1) $\frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}} = \sin A$, read *the sine of A*,

(2) $\frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \cos A$, read *the cosine of A*,

(3) $\frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}} = \tan A$, read *the tangent of A*,

(4) $\frac{b}{a} = \frac{\text{adjacent leg}}{\text{opposite leg}} = \operatorname{ctn} A$, read *the cotangent of A*,

(5) $\frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent leg}} = \sec A$, read *the secant of A*,

(6) $\frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite leg}} = \csc A$, read *the cosecant of A*.

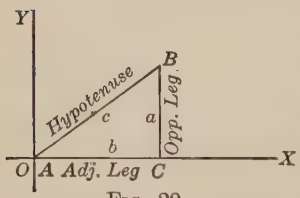


FIG. 20.

These definitions should be memorized, so that they can be used instantly.

34. Simple Relations. The ratios just mentioned depend upon each other in several simple ways. Thus, the last three are *reciprocals* of the first three:

(7) $\frac{b}{a} = 1 \div \frac{a}{b}$, $\frac{c}{b} = 1 \div \frac{b}{c}$, $\frac{c}{a} = 1 \div \frac{a}{c}$.

Hence

(8) $\operatorname{ctn} A = \frac{1}{\tan A}$, $\sec A = \frac{1}{\cos A}$, $\csc A = \frac{1}{\sin A}$.

We often use $1/\cos A$ and $1/\sin A$ instead of using $\sec A$ and $\csc A$. Especially in work with logarithms, $\sec A$ and $\csc A$ are not often used.

Likewise we have

$$(9) \qquad \frac{a}{b} = \frac{a}{c} \div \frac{b}{c}, \qquad \text{and} \qquad \frac{b}{a} = \frac{b}{c} \div \frac{a}{c};$$

hence

$$(10) \qquad \tan A = \frac{\sin A}{\cos A}, \qquad \text{and} \qquad \operatorname{ctn} A = \frac{\cos A}{\sin A}.$$

Finally, since

$$a^2 + b^2 = c^2,$$

we have

$$(11) \qquad \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1, \qquad \text{or} \qquad \sin^2 A + \cos^2 A = 1,$$

$$(12) \qquad \frac{a^2}{b^2} + 1 = \frac{c^2}{b^2}, \qquad \text{or} \qquad \tan^2 A + 1 = \sec^2 A,$$

$$(13) \qquad 1 + \frac{b^2}{a^2} = \frac{c^2}{a^2}, \qquad \text{or} \qquad 1 + \operatorname{ctn}^2 A = \csc^2 A.$$

These relations are often useful. They are called the *Pythagorean* relations, because the relation $a^2 + b^2 = c^2$ was discovered by Pythagoras.

35. The Ratios Depend only on the Angle. The student should understand that for a given angle the values of the six *trigonometric functions* are entirely independent of the lengths of the sides of the right triangle if the angle A is unchanged. Draw any angle A, as in Fig. 21. Choose points B, B', B'', etc., on the terminal side OP. Draw BC, $B'C'$, $B''C''$ perpendicular to OX. Since the triangles are similar, the sides are proportional, and

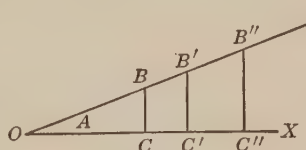


FIG. 21.

$$\frac{BC}{OB} = \frac{B'C'}{OB'} = \frac{B''C''}{OB''} = \sin A,$$

$$\frac{OC}{OB} = \frac{OC'}{OB'} = \frac{OC''}{OB''} = \cos A.$$

That is, these ratios remain unchanged so long as the angle remains unchanged, but they change as the angle changes.

EXAMPLE. Construct a right triangle ABC whose sides are 5, 12, and 13. Find the functions of the angle opposite 5.

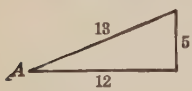


FIG. 22.

SOLUTION. From the definitions, it will be seen that

$$\sin A = 5/13, \quad \cos A = 12/13, \quad \tan A = 5/12,$$
$$\operatorname{ctn} A = 12/5, \quad \sec A = 13/12, \quad \csc A = 13/5.$$

36. Functions of 30°, 45°, and 60°. The angles 30°, 45°, and 60° occur frequently in problems. It is important, and it is fortunately easy, to find their trigonometric functions and to memorize the results.

(*a*) FUNCTIONS OF 45°. Draw an isosceles right triangle ABC. Then angle A = angle B = 45°. Since the relative, and not the actual, lengths of the sides are important, any lengths may be assigned with the condition that the right triangle shall be isosceles. Suppose the length of one side is taken as unity, that is, $a = 1$; then $b = 1$, and

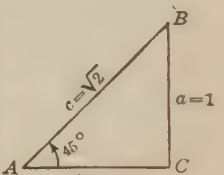


FIG. 23.

$$c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2}.$$

Hence

$$\sin 45° = 1/\sqrt{2} = \sqrt{2}/2, \qquad \tan 45° = 1, \qquad \sec 45° = \sqrt{2},$$
$$\cos 45° = 1/\sqrt{2} = \sqrt{2}/2, \qquad \operatorname{ctn} 45° = 1, \qquad \csc 45° = \sqrt{2}.$$

(*b*) FUNCTIONS OF 30° AND 60°. Construct an equilateral triangle ADB. From the vertex A draw AC, the perpendicular bisector of BD, and consider the right triangle ACB. Angle B = 60°, and angle CAB = 30°.

Now if BC is taken equal to 1, then $BD = BA = 2$, and $b = \sqrt{c^2 - a^2} = \sqrt{4 - 1} = \sqrt{3}$. Hence

$\sin 30° = \frac{1}{2}$, $\quad \sin 60° = \sqrt{3}/2$,

$\cos 30° = \sqrt{3}/2$, $\quad \cos 60° = \frac{1}{2}$,

$\tan 30° = 1/\sqrt{3} = \sqrt{3}/3$, $\quad \tan 60° = \sqrt{3}$,

$\operatorname{ctn} 30° = \sqrt{3}$, $\quad \operatorname{ctn} 60° = 1/\sqrt{3} = \sqrt{3}/3$,

$\sec 30° = 2/\sqrt{3} = 2\sqrt{3}/3$, $\quad \sec 60° = 2$,

$\operatorname{csc} 30° = 2$. $\quad \operatorname{csc} 60° = 2/\sqrt{3} = 2\sqrt{3}/3$.

These results may be put in tabular form for convenience.

Function	30°	45°	60°
sin	$1/2$	$1/\sqrt{2} = \sqrt{2}/2$	$\sqrt{3}/2$
cos	$\sqrt{3}/2$	$1/\sqrt{2} = \sqrt{2}/2$	$1/2$
tan	$1/\sqrt{3} = \sqrt{3}/3$	1	$\sqrt{3}$

The student should visualize the triangles so as to read the results from mental pictures of the diagram.

The cosecant, secant, and cotangent are easily remembered, being the reciprocals of the sine, cosine, and tangent respectively. The functions of angles other than those given above are not so easily computed. The methods of computation are explained in more advanced works on mathematics. The values of the functions of other angles may be obtained from a table.

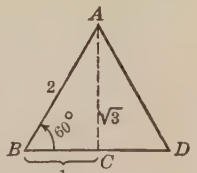


Fig. 24.

Example. Given a right triangle where $A = 30°$, and $a = 50$, to solve the triangle.

Solution.

(1) $30° +$ angle $B = 90°$, hence angle $B = 60°$.

(2) $b/50 = \operatorname{ctn} 30°$, hence $b = 50 \operatorname{ctn} 30° = 50\sqrt{3}$

or

$$b = 50(1.7321) = 86.61.$$

(3) $c/50 = \operatorname{csc} 30°$, hence $c = 50 \operatorname{csc} 30° = 50 \times 2 = 100$.

37. Historical Sketch of Trigonometry. The beginning of trigonometry is found in the Rhind Papyrus (1650 B.C.), now preserved in the British Museum. In it occurs the term *seked*, used in somewhat the same sense as our word cotangent.

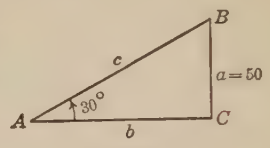


Fig. 25.

The principal epochs in the development of trigonometry are:

1. Greek (on the Island of Rhodes and at Alexandria), 150 B.C.–200 A.D. The most famous Greek contributors were Hipparchus (circa 150 B.C.), who used the chord subtended by the double angle, and Ptolemy (87 A.D. to 165 A.D.), who gave a table of chords for every half-degree of the quadrant.

2. Hindu, 450 A.D. to 1100 A.D. Among the famous Hindus were Aryabhatta (476 A.D.), who made a table of sines at intervals of 3° 45′, and Baskara (1114 A.D.), who calculated a table of sines for every degree.

3. Arab (in western Asia and Spain), 675 A.D.–1200 A.D. The Arabs translated and used the works of Aryabhatta and Ptolemy.

4. European, 1200 A.D. Johann Müller (called also Regiomontanus) (1436–1476 A.D.) published the first work ("De Triangulis"), which superseded the works described above. Euler (1707–1783 A.D.), a Swiss, whose life was spent largely in St. Petersburg and Berlin, contributed much to the establishment of trigonometry as an independent science.

38. Geometric Principles. The student will find it worth while to keep in mind the following principles of plane geometry.

1. *The sum of the interior angles of a triangle is equal to 180°.*

2. *The sum of the two acute angles of a right triangle is equal to 90°, and either acute angle is the complement of the other.*

3. *In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.*

4. *In any triangle the greater angle is opposite the greater side, and conversely.*

EXERCISES

1. The values a, b, c in each of the following exercises are the lengths of the sides of a right triangle, the side c being in each case the hypotenuse. Draw the triangle and then find the trigonometric functions of angle A.

(*a*) $a = 3, b = 4, c = 5.$
(*b*) $a = 8, b = 15, c = 17.$
(*c*) $a = 9, b = 40, c = 41.$
(*d*) $a = 24, b = 10, c = 26.$
(*e*) $a = 12, b = 35, c = 37.$
(*f*) $a = 7, b = 24, c = 25.$

2. In each of the following exercises, two sides of a right triangle are given, and c denotes the hypotenuse of the right triangle. Find the side not given and read the functions of angle A. Also read the functions of angle B.

(*a*) $a = 21, b = 20, c = ?$
(*b*) $a = 24, b = 32, c = ?$
(*c*) $a = 36, c = 45, b = ?$
(*d*) $a = 63, c = 65, b = ?$

3. In each of the following right triangles find the remaining parts, and the functions of angle A. Read the functions of angle B.

(*a*) $A = 30°, a = 4.$
(*b*) $A = 45°, a = 15.$
(*c*) $A = 60°, b = 20.$
(*d*) $A = 30°, c = 10.$
(*e*) $A = 60°, b = 18.$
(*f*) $A = 45°, c = 18.$

4. Show that in a right triangle ABC:

$a = c \sin A.$ $\quad a = b \tan A.$ $\quad c = b \sec A.$
$b = c \cos A.$ $\quad b = a \operatorname{ctn} A.$ $\quad c = a \csc A.$

5. Draw a right triangle for each of the following exercises. Compute the value of the third side, and find the values of the five functions not given.

(*a*) $\sin A = 3/5.$
(*b*) $\cos A = 2/5.$
(*c*) $\tan A = 3/8.$
(*d*) $\operatorname{ctn} A = 5/7.$
(*e*) $\sec A = 10/7.$
(*f*) $\csc A = 2.$
(*g*) $\sin A = 1/2.$
(*h*) $\cos A = 1/2.$
(*i*) $\tan A = 5/10.$

6. Two miles from a straight road is a fort. In the fort is a gun whose range is 3.8 miles. By means of a scale drawing, find what distance along the road is commanded by the gun.

7. From a point 180 ft. from the foot of a wireless telegraph mast, the top of the mast has an angle of elevation (see § 39) of 30°. Find the height of the mast.

8. In order to determine the width of a river, the distance between two points A and B, close to the river bank, was measured and found to be 400 ft. The angles BAP and ABP (the point P being on the opposite bank) were also measured and found to be 40° and 60° respectively. Find, approximately, the width of the river. Use graphical method.

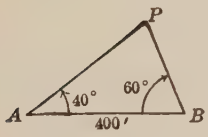


Fig. 26.

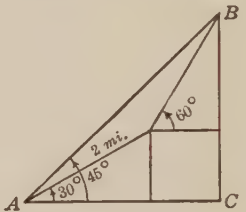


Fig. 27.

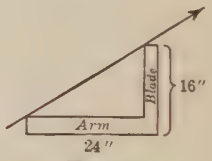


Fig. 28.

9. Two sides of a triangle are found to be 80 rods and 96 rods, and the included angle 60°. Find from a drawing the remaining parts.

10. From a point A at the foot of a mountain, the angle of elevation of the summit (see § 39) is found to be 45°; after walking two miles toward the summit on an incline of 30°, the angle of elevation is then found to be 60°. Find the height of the mountain.

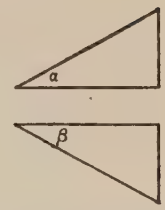
11. A carpenter's square is 16″ by 24″; the blade is $1\frac{1}{2}$″ wide, and the arm is 2″ wide. A man places the square on the top of a post 4 ft. high so that the end of the arm is 36 ft. from a building. The arm is horizontal and the blade vertical. How high is the building, if the line of sight over the arm and blade, as shown in the figure, just intersects the top of the building?

39. Terms. In verbal problems certain terms are used with technical meanings.

(*a*) Angles of Elevation and Depression. The *angle of elevation*, or the *angle of depression*, of an object is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane. In Fig. 29, α is an angle of elevation, and β is an angle of depression.

(*b*) ALTITUDE OF A HEAVENLY BODY. The angle of elevation of a heavenly body is called its *altitude.*

(*c*) BEARING. The direction shown by the compass is called the *bearing*; thus N. 30° E. means the direction reached after sighting north and then swinging 30° toward the east.



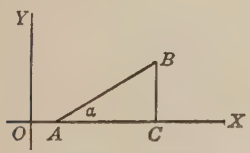
α = Angle of Elevation
β = Angle of Depression

FIG. 29.

(*d*) INCLINATION. The angle which a line makes with the positive direction of the x-axis when the line is given the upward direction is called the *inclination of the line*, or simply the *inclination*, and is frequently designated by α (alpha).

NOTE. For ordinary purposes angles can be measured with a protractor to a degree of accuracy of about 30′. For work out of doors it is customary to use a transit, an instrument by means of which angles can be measured to half minutes or less.

(*e*) GRADE, OR SLOPE. The *grade*, or *slope*, of a straight line AB (Fig. 30) is defined as the tangent of its inclination. The angle is frequently denoted by α and the slope by m. The slope or grade $= m = \tan \alpha$. If a straight line rises 18 feet in a horizontal distance of 100 feet, the grade, or slope, is said to be 18 in 100 or 0.18. Grades are often expressed in percent. A 6 percent grade has a



Grade or Slope

$\frac{CB}{AC} = \tan \alpha = m.$

FIG. 30.

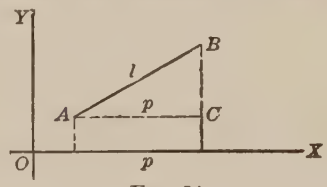


FIG. 31.

vertical rise of 6 feet in a horizontal distance of 100 feet. The inclination of this line is about $3\frac{1}{2}°$, since $\tan 3\frac{1}{2}° = .06$, nearly. The slope of a horizontal line is zero.

There is no such thing as the slope of a vertical line, but it is sometimes said that the slope of a vertical line is infinite. This is discussed more fully on page 69.

(*f*) Projection. The distance between the feet of the perpendiculars dropped from the ends of a line-segment l on another line OX is called the *projection* of l on OX. The length of the projection p is found as follows:

$$\frac{p}{l} = \cos A, \qquad \text{or} \qquad p = l \cos A.$$

That is, the projection of a line-segment l upon another line is equal to l times the cosine of the angle included between the segments, produced if necessary.

40. Arrangement and Use of Tables. The values of the trigonometric functions, correct to four decimal places, for every degree from 0° to 90° have been computed, and tables giving these values have been printed. Such a table is shown on page 387 and will suffice for the work of the next few pages. The student should understand that the values given are only approximate. They are correct to the nearest ten thousandth. In finding the functions of angles from 0° to 45° read from the top down; in finding the functions of angles from 45° to 90° read from the bottom up.

Example. A rope 72 feet long reaches from the top of a flag pole to a stake on the same horizontal level as the foot of the pole. If the rope when taut makes the angle of 37° 20′ with the horizontal, find the remaining parts of the triangle.

Solution.

$$\begin{aligned} B &= 90° - A \\ &= 90° - 37° 20' \\ &= 52° 40'. \end{aligned}$$

To find a,

$$a/c = \sin A.$$

Then

$$a = c \sin A.$$
$$= 72\ (.6064)$$
$$= 43.66 \text{ ft.}$$

To find b,

$$b/c = \cos A.$$

Then

$$b = c \cos A.$$
$$= 72\ (.7951)$$
$$= 57.25 \text{ ft.}$$

CHECK. To check the solution use the equation

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$
$$a = 43.66, \quad b = 57.25, \quad c = 72.$$
$$c + b = 129.25, \quad c - b = 14.75.$$
$$a^2 = 1906.20, \quad (c + b)(c - b) = 1906.44.$$

Hence $a^2 = c^2 - b^2$, nearly.

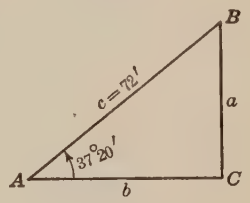


FIG. 32.

Since the exact value of angle A could not be found in the table on p. 387 it was necessary to interpolate by using proportional parts.

$$\sin 38° = .6157$$
$$\sin 37° = .6018$$

Difference for $1° = .0139$

$20' = \frac{1}{3}$ of $1°$, $\frac{1}{3}$ of $.0139 = .0046$

Hence

$$\sin 37° 20' = .6018 + .0046$$
$$= .6064$$

The figure is drawn to enable the student to visualize the conditions more readily. A high degree of accuracy in the drawing is not so essential as in scale drawings, since the results are obtained independently of the drawing.

41. Functions of Complementary Angles. In Fig. 20, it is readily seen that angle B is the complement of angle A, that is, $B = 90° - A$. Hence we may write

$$\sin A = \frac{a}{c} = \cos B, \qquad \operatorname{ctn} A = \frac{b}{a} = \tan B,$$

$$\cos A = \frac{b}{c} = \sin B, \qquad \sec A = \frac{c}{b} = \csc B,$$

$$\tan A = \frac{a}{b} = \operatorname{ctn} B, \qquad \csc A = \frac{c}{a} = \sec B.$$

NOTE. The sine and cosine are called co-functions of each other. Similarly tangent and cotangent, secant and cosecant are co-functions. Hence

$$\sin 72° = \cos (90° - 72°) = \cos 18°.$$
$$\cos 85° = \sin (90° - 85°) = \sin 5°,$$

and so on. Any function of an angle between 45° and 90° may be found by taking the co-named function of the complementary angle, which is between 0° and 45°.

THEOREM. *A function of an acute angle is equal to the co-function of its complementary angle.*

42. General Directions for Solving Right Triangles.

1. Make a drawing representing the given conditions.

2. If one acute angle is known, subtract it from 90° and get the other.

3. In solving for a side, write the fraction of which the unknown side is the numerator, and the known side the denominator. Then select the proper function of the angle. Thus, if in a right triangle ABC, $c = 50$ and $\angle A = 28°$, we have $B = 90° - 28° = 62°$; and $a/50 = \sin 28°$, or $a = 50 \sin 28°$.

4. Check the results by determining whether they satisfy the following conditions:

$$A + B = 90°, \quad a^2 + b^2 = c^2, \quad \text{or} \quad a^2 = (c + b)(c - b).$$

EXERCISES

1. The inclination of a hill is 15°. If a man walks one-fourth mile up the hill, how many feet has he risen?

2. A guy wire known to be 306 ft. long extends from the top of a radio mast to an anchor on the level ground. If the rope makes an angle of 38° with the horizontal, how far is the anchor from the foot of the mast? How high is the mast?

3. Mt. Hood is at a distance of 51 miles from Portland, Oregon, in a direction 14° south of east. How far south is it, and how far east is it from the city?

4. The angle of elevation of the top of a tree, at a point 108 ft. from the foot of the tree measured on level ground, is 31°. How high is the tree?

5. The angle of elevation of the sun is 42°, and the length of a tree's shadow on a level plane is 54 ft. Find the height of the tree.

6. A chord of a circle is 18.6 ft. The angle which it subtends at the center of the circle is 48°. Find the radius of the circle.

7. A tree standing on the bank of a river is known to be 54 feet high. From the opposite bank the angle of elevation of the top of the tree, measured from a point 4 ft. 6 in. above the foot of the tree, is found to be 16°. Find the width of the stream.

8. Certain lots in a city are laid out by lines perpendicular to B St. and running through to A St., as shown in the figure. Required the frontage of each lot on A St., if the angle between the streets is 36° 30′.

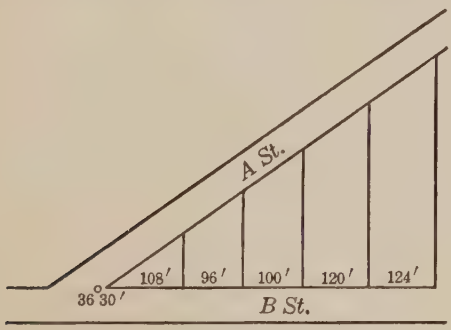


FIG. 33.

9. Wishing to measure the length of a pond, a class constructed a right triangle as shown in Fig. 34. If $AB = 833$ ft. and $A = 28° 24'$, find the distance AC.

10. From where I stand, a tree 44 ft. away has an angle of elevation of 42° 10′. From the same point another tree 73 ft. distant has an angle of elevation of 27°. Which tree is the taller and how much?

11. A sewer of tile is laid 5 ft. below the surface of the ground at the lower end and 7 ft. at the upper end. If the sewer is $\frac{1}{2}$ mile long and the

surface of the ground has a uniform inclination of 1°, how many feet of fall does the sewer have between the two ends?

12. A hoisting crane has an arm 24 feet long. If the arm makes an angle of x degrees with the horizontal line AC, find the distances AC and BC when $x = 15°$; when $x = 40°$; when $x = 72°$.

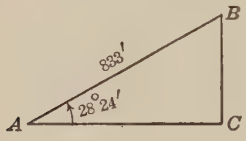


Fig. 34.

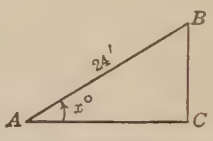


Fig. 35.

13. Taking the radius of the earth equal to 3960 miles, find (*a*) the radius of the parallel at latitude 66° north, (*b*) the length of the parallel at latitude 23° south.

14. The base of the great pyramid of Gizeh is a square whose side is 756 feet, and its top is a square whose side is 32 feet. Each face makes an angle of 52° nearly with the horizontal plane. Find its height.

15. A road leading up a hillside is 602 feet long and has an inclination of 5°. It is desired to cut the road to a 6% grade. What will be the vertical cut at each 100 ft. mark measured along the horizontal?

16. To determine the width of a stream a surveyor measures a line, AC, 400 ft. long, on one bank. At C he turns a right angle and his assistant places a stake B at the point where the perpendicular to AC meets the opposite bank. The angle CAB measures 48° 10′. How wide is the stream?

17. Solve the following right triangles ($C = 90°$):

Given Parts

(*a*) $A = 35°$, $c = 80$.	(*d*) $A = 68°$, $a = 48.5$.
(*b*) $B = 62°$, $c = 72$.	(*e*) $B = 84°$, $a = 100$.
(*c*) $A = 37°$, $b = 96$.	(*f*) $A = 76°$, $c = 65$.

18. The carpenter's rule for constructing an angle equal to $\frac{3}{4}$ of a right angle is to construct a right angle whose sides are 5 inches and 12 inches, respectively, and then take the greater acute angle in the triangle. How large is the error?

19. A man climbs a hill, the inclination of which is 28°. His pocket barometer shows that at the end of three hours he has increased his elevation 5110 ft. Find his average speed up the slope.

20. A barn 36 ft. wide has a gambrel roof. The lower rafters make an angle of 60° with the horizontal. The upper rafter makes an angle of 60° with the vertical. The upper and lower rafters are equal in length. Find the length of the rafters. See Fig. 36.

21. Draw a right triangle and denote the sides and angles in conventional notation. Derive formulas for the area when:

(*a*) one leg and the adjacent angle are known.
(*b*) the hypotenuse and an adjacent angle are known.

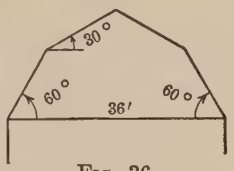


FIG. 36.

22. The Washington Monument is 555 feet high. At a certain point on the same horizontal level as the base, the angle of elevation of the top is 28°. Find the distance of the point from the foot of the monument.

43. To Find Angles. It is frequently necessary to find an acute angle of a right triangle, the numerical value of one of its functions being given. We then look for the given number in the column having the given function as a caption at the top (if the angle is between 0° and 45°), or at the bottom (if the angle is between 45° and 90°). If we find it in the column having the given function as a top caption, the required angle will be found in the same horizontal line and in the left-hand column. If the given function is a bottom caption, the desired angle will be found in the right-hand column.

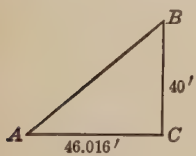


FIG. 37.

EXAMPLE. Let it be required to find the angle of elevation of the sun, if a pole 40 ft. high casts a shadow 46.016 feet long.

SOLUTION.

$$\tan A = \frac{40}{46.016} = .8693$$

From the table, angle A is seen to be 41°.

44. Interpolation. As long as the angle is given in degrees, the table on page 387 is sufficient. But when the angle is

expressed in degrees, minutes, and seconds, the table does not give the value of the function directly. It is then necessary to resort to a process called interpolation. Briefly expressed, in the process of interpolation, we assume that sin 31° 30′ is found by adding to sin 31° half the difference between sin 31° and sin 32°. Likewise sin 35° 50′ is found by adding to sin 35° five-sixths of the difference between sin 35° and sin 36°. In general this process gives results that are only approximately true. Graphically, interpolation is made as if the curves $y = \sin x$, $y = \tan x$, $y = \log x$, etc., consisted of straight line segments. However, for angles in which the changes are very small, interpolation gives results which are correct to the number of decimal places given in the table.

EXAMPLE 1. Find sin 18° 24′.

$$\begin{aligned} \sin 19^\circ &= .3256 \\ \sin 18^\circ &= .3090 \\ \hline \text{Difference for } 1^\circ \text{ or } 60' &= .0166 \\ \text{Difference for } 1' \text{ is } 1/60 \text{ of } .0166 &= .00027+ \\ \text{Difference for } 24' \text{ is } 24 \times .00027+ &= .0066+ \end{aligned}$$

Adding this to sin 18°, we have:

$$\sin 18^\circ\ 24' = .3156$$

EXAMPLE 2. Find x, if $\tan x = .3163$.

Looking in the table for the tangent that is a little less than .3163 and for the next larger value of the tangent, we have

$$\begin{aligned} \tan 18^\circ &= .3249 \\ \tan 17^\circ &= .3057 \\ \hline & .0192, \text{ tabular difference for } 1^\circ. \end{aligned}$$

Therefore x lies between 17° and 18°. Furthermore

$$\begin{aligned} \tan x &= .3163 \\ \tan 17^\circ &= .3057 \\ \hline & .0106 \text{ difference.} \end{aligned}$$

The correction is $\frac{106}{192}$ of 60′, or 33′ +. Hence $x = 17^\circ\ 33' +$.

45. Accuracy. The degree of accuracy obtained by using trigonometric tables correct to four decimal places is sufficient for many ordinary applications. If the problem involves extended land surveys, measurements in astronomy, and other kinds of work in which instruments of precision are used, five-place tables are to be preferred. Tables correct to more decimal places are obtainable if the accuracy of the data warrants their use.*

In arithmetic, algebra, and geometry, the data are usually assumed to be correct. In practical problems in trigonometry, the data frequently depend on *measurements*, which are always subject to error. It is essential, therefore, that each measurement have approximately the same degree of accuracy as shown by the number of significant figures. Suppose the measurements of a rectangle are given as approximately 8 ft. wide and 425 ft. long. An error of 0.1 ft. in the width means an error of 42.5 sq. ft. in the area. An error of 0.1 ft. in the length means an error of only 0.8 sq. ft. in the area. One error is 50 times as great as the other. These measurements would not be called consistent.

The following directions will aid the student in making consistent measurements and avoiding useless calculations.

1. All measured or calculated distances should show the same number of *significant figures*, as $l = 425$, $w = 8.26$.

2. If the distances show *two* significant figures, the angles should be obtained to the nearest *half-degree*.

3. If the distances show *three* significant figures, the angles should be obtained to the nearest *minute*.

4. If the distances show *four* significant figures, the angles should be obtained to the nearest *10 seconds*.

5. If the distances show *five* significant figures, the angles should be obtained to the nearest *second*.

* See, for example, The Macmillan Tables.

EXERCISES

1. The Eiffel Tower is 984 feet high. What is the angle of elevation of its top, when viewed from a distance of 6 000 feet?

2. A railway to the top of Pike's Peak rises 7 800 feet in a distance of $9\frac{1}{4}$ miles along the track. At what angle would the track ascend if the inclination were uniform?

3. A carpenter's square BCA, held against the vertical stick BD, rests on a sloping roof AD as shown in figure. It is found that $AC = 24$ in., and $CD = 8.4$ in. Find the inclination of the roof.

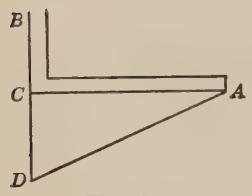


FIG. 38.

4. Rain drops are falling vertically with a velocity of 40 feet per second. At what angle would they appear to fall to a man running at the rate of $7\frac{1}{2}$ miles per hour? To a man in an automobile moving 20 miles per hour? To a man in an express train moving at the rate of 60 miles per hour?

5. The Grand Canyon of the Colorado River has in places a depth of one mile and a width of 8 miles from rim to rim. Assuming the banks are equally inclined planes intersecting at the bottom of the canyon, find the inclination of the banks with the horizontal.

6. When light passes from one medium into another, the angles of incidence and refraction (I and R) are related thus:

$$\sin R = k \sin I.$$

If $k = \frac{5}{8}$ and $I = 22° 30'$, find R.

7. In the Panama Canal, the Gatun Dam has three different slopes: the ratio of the horizontal to the vertical near the base is 16 to 1; near the middle of the cut (vertically) the ratio is 8 to 1; at the top the ratio is 4 to 1. What three different angles does the surface of the dam make with the horizontal?

8. On the steepest part of the Mt. Washington railway (Jacob's Ladder) there is a rise of $13\frac{1}{2}$ inches for every 3 feet of track. What angle does the track make with the horizontal? At this rate what would be the rise in a distance of one mile along the track?

9. The legs of a pair of dividers $5\frac{1}{2}$ inches long are set at an angle of $28° 40'$. Determine the distance between the points.

10. Solve the following right triangles ($C = 90°$):

	Given Parts		Required Parts		
(*a*)	$a = 60$,	$c = 100$,	$A = 36° 52'$,	$B = 53° 8'$,	$b = 80$.
(*b*)	$a = 147$,	$c = 184$,	$A = 53° 2'$,	$B = 36° 58'$,	$b = 110.66$
(*c*)	$a = 50$,	$b = 60$,	$A =$,	$B =$,	$c =$.
(*d*)	$a = 38$,	$c = 72$,	$A =$,	$B =$,	$b =$.
(*e*)	$a = 45$,	$c = 65$,	$B =$,	$A =$,	$b =$.

11. The eye sees colors incorrectly at certain angles from the center of the field of vision, different for different persons. In one experiment red appeared as yellow when 14.3 cm. ahead and 8.3 cm. to the right, and black when 15 cm. to the right. Find the angle from the central line of vision in each case.

12. In planning a truss for a bridge it is necessary to have the upright BC made 12 ft. long, and the horizontal AC $7\frac{1}{2}$ ft., as shown in the figure. What angle does AB make with AC? With BC?

13. The arm of a derrick for hoisting coal is 22′ 8″ long. It swings over an opening 18′ in radius. How far is the top of the arm above the foot? Find the angle of elevation.

14. The average inclination of the bed of a mountain stream is 2° 48′. Find approximately its fall in a distance of 4 miles along the stream.

15. To find the length of a pond a group of students laid off the right-triangle ACB. They found $AC = 396'$, and angle $A = 41°$. Find the length of the pond AB.

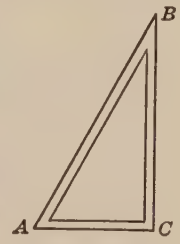


FIG. 39.

16. If a roadway rises uniformly 9 feet in a horizontal advance of 40 feet, what is the difference in elevation of two points 2754 feet apart if the distance is measured along the surface of the roadway?

17. The summit of a mountain, known to be 12 480 feet high, is seen at an angle of elevation of 22° 30′ from a camp whose altitude is 5840 feet. Compute the air-line distance from the camp to the summit of the mountain.

CHAPTER III

EXPONENTS — LOGARITHMS

46. Integral Exponents. When two or more factors are alike, it is often convenient to indicate the fact as follows:

$$2 \times 5 \times 5 \times 5 = 2 \times 5^3,$$
$$3 \times a \times a \times a \times b \times b = 3a^3b^2,$$
$$a \times a \times a \times a \times a \ldots \text{(to } n \text{ factors)} = a^n.$$

The number a is called the *base*, and the figure (n) written to the right and above is called the *exponent*. The fundamental laws of exponents may be stated as follows:

I. Law of Multiplication.

$$a^m \times a^n = a^{m+n}.$$

This is known as the *index law.*

II. Law of Division.

$$a^m \div a^n = a^{m-n}.$$

III. Law of Involution.

$$(a^m)^n = (a^n)^m = a^{mn}.$$

In elementary algebra these formulas are proved true only for integral exponents, but it is often convenient to use other exponents. Hence the above laws must be extended so as to include (a) fractional exponents, (b) negative exponents.

47. Fractional Exponents. The meaning of exponents may be so defined that $a^m \times a^n = a^{m+n}$ is true whatever values m and n may have, at least when a, a^m, and a^n are positive.

Thus $a^{1/2}$ is a number such that

$$a^{1/2} \times a^{1/2} = a^{1/2+1/2} = a^1.$$

That is, $a^{1/2}$ is the square root of a, written $\sqrt{a}$. Likewise,

$$a^{3/4} \times a^{3/4} \times a^{3/4} \times a^{3/4} = a^3.$$

That is, $a^{3/4}$ is the fourth root of a^3, written $\sqrt[4]{a^3}$. In general

$$a^{p/q} \times a^{p/q} \times a^{p/q} \ldots (q \text{ factors}) = a^p,$$

or

$$a^{p/q} = \sqrt[q]{a^p}.$$

In fractional exponents, the numerator indicates the power, and the denominator indicates the root.

48. Zero Exponents. To find a meaning for a^0, we assume that the index law holds for all exponents. Then we have

$$a^0 a^n = a^{0+n} = a^n, \qquad \text{or} \qquad a^0 = a^n \div a^n = 1.$$

Any number with an exponent zero gives the result 1.

49. Negative Exponents. To find a meaning for a^{-n} (n positive), assume the index law holds. Then

$$a^n \times a^{-n} = a^0 = 1.$$

Dividing both sides by a^n, we obtain

$$a^{-n} = \frac{1}{a^n}.$$

Dividing both sides by a^{-n}, we obtain

$$a^n = \frac{1}{a^{-n}}.$$

Hence, a factor may be changed from the numerator to the denominator of a fraction, or vice versa, by changing the

sign of its exponent. For example

$$6^{-2} = \frac{1}{6^2}, \qquad \left(\frac{1}{9}\right)^{-1/2} = \frac{1}{9^{-1/2}} = 9^{1/2} = 3.$$

Also

$$a^{-2} = \frac{1}{a^2}, \qquad a^{-1/2} = \frac{1}{a^{1/2}}, \qquad \frac{1}{a^{-1/3}} = a^{1/3}, \qquad \frac{1}{a^{-2}} = a^2.$$

50. Notation of Science. In scientific work, very large or very small numbers are often expressed by the aid of exponents, as 2.7×10^8 or 3.4×10^{-6}, instead of being written out in full, thereby indicating also the order of known accuracy.

Thus $3\,670\,000 = 3.67 \times 10^6$, and $2.41 \times 10^{-3} = 0.00241$

EXERCISES

1. Find the numerical value of:

(*a*) 3^4. (*e*) $25^{1/2}$. (*i*) $36^{1/2}$. (*m*) $4^{1/2} \times 125^{1/3}$.

(*b*) 2^5. (*f*) $27^{1/3}$. (*j*) $16^{1/4}$. (*n*) $2^3 \times 3^5 \times 8^{2/3}$.

(*c*) 12^2. (*g*) $64^{2/3}$. (*k*) $\sqrt[4]{81^3}$. (*o*) $2^0 \times 3^5 \times 7^0$.

(*d*) $\sqrt{16}$. (*h*) $32^{1/5}$. (*l*) $9^{1/2}$. (*p*) $(a^m \times a^n)^3$.

2. Simplify:

(*a*) $27^{2/3}$. (*e*) $a^4 \times a^{-4}$. (*i*) $\sqrt[3]{a^6}$.

(*b*) $a^4 \times a^{-2}$. (*f*) $64^{5/6}$. (*j*) $\sqrt[4]{a^{12}}$.

(*c*) $b^5 \times b^{-5}$. (*g*) 8^0. (*k*) $\sqrt[2]{a^{10}}$.

(*d*) $a^3 \times a^{-7}$. (*h*) $\dfrac{1}{2^{-3}}$. (*l*) $a^2 \times a^{-2}$.

3. Why do we define a^0 as equal to 1?

4. Translate each of the fundamental laws into words.

5. Express in a form free from negative and zero exponents:

(*a*) $5x^{-2} + 6x^{-1} + 3x^0$. (*c*) $\dfrac{7x + 4x^0}{x^{-3}} + 3x^{-2}$.

(*b*) $\dfrac{x^2 + 5x + 7}{x^{-2}}$. (*d*) $3x^{-2} + 5x^{-1} + 7$.

6. Find the value of x in each of the equations:

(*a*) $2^x = 64$. (*b*) $10^x = 1000$. (*c*) $10^x = .001$.

(*d*) $3^x = 27$. (*e*) $10^x = 1$. (*f*) $2^x = \frac{1}{4}$.

(*g*) $25^x = 5$. (*h*) $10^x = .01$. (*i*) $3^x = \frac{1}{81}$.

7. The distance from the sun to the earth is 9.29×10^7 miles; from the sun to Neptune 2.79×10^9 miles; from the sun to the nearest star 2.5×10^{13} miles. Express these numbers without using exponents.

8. Translate into the usual notation:

(*a*) 4.24×10^9. (*b*) 3.2×10^{10}. (*c*) 7.8×10^{12}.

(*d*) The diameter of an "H" molecule is 5.8×10^{-8} cm.

(*e*) The weight of an "H" molecule is 4.6×10^{-24} gm.

(*f*) The wave length of sodium light is 5.89×10^{-5} cm.

(*g*) The distance to the star Polaris is 4.67×10^{14} miles.

(*h*) Light travels in one second 2.999×10^8 meters.

9. Express in the notation of science:

(*a*) The distance to the moon is 240 000 miles.

(*b*) The energy required to raise the temperature of 1 gram of water through 1° C. is 41 600 000 ergs.

(*c*) One kilo-watt equals 10 000 000 000 ergs.

(*d*) One mile equals 161 000 cm.

(*e*) One day equals 86 400 sec.

51. Historical Note. Cajori says: "The miraculous powers of modern calculation are due to three inventions: the Hindu Notation, Decimal Fractions, and Logarithms." The invention of logarithms was admirably timed, for Kepler was then examining planetary orbits, and Galileo had just turned his telescope to the stars. Logarithms were invented early in the seventeenth century (1614), by John Napier, a wealthy Scotch baron who took an active part in the political and religious controversies of his day, but studied mathematics and science for amusement. He soon saw the desirability of aiding his contemporaries by devising some method of shortening the processes of multiplication, division, extracting roots, and raising numbers to powers.

The first logarithmic tables appeared in 1614, and met with an immediate appreciation both in England and on

the continent. Henry Briggs, Professor of Geometry in Gresham College, London, suggested the use of 10 as a base.

As a time-saving device, there is probably no mathematical invention or discovery comparable to logarithms, except the Hindu-Arabic notation. DeMorgan says that by the aid of logarithms the labor of computing has been reduced for mathematicians to about one-tenth part of the previous expense of time and labor. Laplace, the great French mathematician and astronomer, said: "The method of logarithms, by reducing to a few days the labors of many months, doubles, as it were, the life of the astronomer, besides freeing him from the errors and disgust inseparable from long calculation."

52. Logarithms. The ***logarithm*** of a number is the exponent to which the ***base*** must be raised to produce the given number.

For example, since $10^2 = 100$, 2 = the logarithm of 100 to the base 10. Likewise, since $10^3 = 1000$, 3 = the logarithm of 1000 to the base 10.

In general, if $b^x = N$, x = the logarithm of N to the base b. The student should understand clearly that *a logarithm is an exponent*, and *a table of logarithms is a table of exponents*. In the equation $10^2 = 100$, the 2 can be thought of as the exponent 10 must have to produce 100, or as the logarithm of 100 to the base 10, that is, $\log_{10} 100 = 2$.

The statements in the following table are equivalent.

Exponential Form	Logarithmic Form
$10^3 = 1000$,	$\log_{10} 1000 = 3$.
$10^2 = 100$,	" $100 = 2$.
$10^1 = 10$,	" $10 = 1$.
$10^0 = 1$,	" $1 = 0$.

Exponential Form	Logarithmic Form
$10^{-1} = 0.1$,	$\log_{10} 0.1 = -1$.
$10^{-2} = 0.01$,	" $0.01 = -2$.
$10^{-3} = 0.001$,	" $0.001 = -3$.
$10^{-4} = 0.0001$,	" $0.0001 = -4$.

Note. For the *logarithm of* N it is customary to write $\log N$. If we wish to specify $\log N$ to the base b, we may write $\log_b N$, read "logarithm of N to the base b."

53. Base. Any positive number, except 1, may be taken as the base of a system of logarithms. The base 10 is largely used in numerical calculations, and the base $e = 2.71828\ldots$ in theoretical considerations. Logarithms to the base 10 are most convenient in computation since with this base the integral part of the logarithm may be determined by inspection. The use of logarithms is easily learned and readily applied in finding products, quotients, roots and powers.

54. Use of a Logarithm Table. Since most numbers are not integral powers of 10, logarithms will in general consist of an integral part and a decimal part. If we desire to find what exponent 10 must have to equal (say) 634, we consult a table. Since 634 is greater than 100 and less than 1000, the power to which 10 must be raised to produce 634 will be between 2 and 3, that is, 2 plus some decimal. The decimal part is given in the table. In the table look for 63 in the column headed N (number). To the right of 63 in the column headed 4 is 80209. Thus $\log_{10} 634 = 2.80209$, which means that $10^{2.80209} = 634$, approximately.

The integral part of a logarithm is called its ***characteristic*** and the decimal part is called its ***mantissa.*** A table of logarithms is a table of mantissas.

(*a*) The Characteristic. From § 52 it will be seen that the characteristic for a number between 10^0 and 10^1 is 0; between 10^1 and 10^2 is 1; between 10^2 and 10^3 is 2, etc.

Statement. *The characteristic for a number greater than 1 is positive, and one less than the number of digits in the given number to the left of the decimal point.*

For example, since

8.4 lies between 10^0 and 10^1, $\log 8.4 = 0 +$ some mantissa;

74. lies between 10^1 and 10^2, $\log 74 = 1 +$ some mantissa;

375.8 lies between 10^2 and 10^3, $\log 375.8 = 2 +$ some mantissa.

NOTE. The student should see that the characteristic depends only on the location of the decimal point, rather than upon the particular digits involved.

The characteristic for a positive number less than unity is a negative integer whose absolute value is one more than the number of zeros between the decimal point and the first significant figure. For example:

$$\log 0.600 = -1. + \text{the mantissa } .77815$$

which is written in the form

$$\log 0.600 = \bar{1}.77815, \quad \text{or} \quad 9.77815-10;$$

and

$$\log 0.0034 = -3. + \text{the mantissa } .53148$$

which is written in the form

$$\log 0.0034 = \bar{3}.53148, \quad \text{or} \quad 7.53148-10.$$

In these forms of writing, *only the characteristic is negative.* For convenience the mantissa is always taken as positive. The characteristic may be either positive or negative.

(*b*) MANTISSA. The mantissa is determined by the value and the order of the digits in the number. It is independent of the decimal point. Thus

$$10^{0.3304} = 10^0 \times 10^{.3304} = 2.14; \text{ hence } \log 2.14 = 0.3304$$
$$10^{1.3304} = 10^1 \times 10^{.3304} = 21.4; \text{ hence } \log 21.4 = 1.3304$$
$$10^{2.3304} = 10^2 \times 10^{.3304} = 214.; \text{ hence } \log 214. = 2.3304$$

EXERCISES

1. From the table find the logarithm of:

(*a*) 1623 (*c*) 81.06 (*e*) 375.8 (*g*) .02375
(*b*) 7245 (*d*) 217.3 (*f*) 2.842 (*h*) 0.04257

2. From the table find the number whose logarithm is:

(*a*) 1.36568 (*c*) 0.70509 (*e*) $\bar{2}.56679$
(*b*) 2.84073 (*d*) 0.36511 (*f*) $\bar{1}.69161$

3. Find from a table of logarithms of trigonometric functions:

(*a*) log sin 18°. (*c*) log tan 41°. (*e*) log cos 62°.
(*b*) log cos 33°. (*d*) log sin 54°. (*f*) log tan 68°.

4. From a table of logarithms of trigonometric functions, find the acute angle x, having given:

(a) $\log \sin x = 9.64184$ (d) $\log \sin x = 9.80807$

(b) $\log \cos x = 9.98060$ (e) $\log \cos x = 9.98690$

(c) $\log \tan x = 9.64858$ (f) $\log \tan x = 9.90837$

55. Laws of Exponents. The laws of exponents in algebra as discussed in this chapter are:

(a) $a^n \times a^m = a^{n+m}$. (b) $a^n \div a^m = a^{n-m}$.

(c) $(a^n)^m = a^{nm}$. (d) $\sqrt[m]{a^n} = a^{n/m}$.

(e) $a^n \times a^{-n} = a^0 = 1$. (f) $a^{-n} = \dfrac{1}{a^n}$.

Since logarithms are exponents, every theorem about exponents is also a theorem about logarithms, when expressed in the language of logarithms. In the following four sections the above laws should be borne in mind.

56. Logarithm of a Product. *The logarithm of a product is found by adding the logarithms of its factors.* Thus, if $P = MN$, then $\log_{10} P = \log_{10} M + \log_{10} N$.

EXAMPLE. Multiply 1.72 by 31.6

	$\log_{10} 1.72$	$= 0.23553$
	$\log_{10} 31.6$	$= 1.49969$
Then	$\log_{10}$ (Prod.)	$= 1.73522$
	Prod.	$= 54.353$

PROOF. Let M and N be the factors and x and y their respective logarithms. Then taking 10 as the base and remembering that $x = \log_{10} M$ and $y = \log_{10} N$, we have:

$$M = 10^x, \quad \text{and} \quad N = 10^y.$$

Multiplying, we find

$$MN = 10^{x+y}.$$

Therefore

$$\log_{10} MN = x + y = \log_{10} M + \log_{10} N.$$

NOTE. The proofs given here are for base 10 only. The proof for the base b is similar.

57. Logarithm of a Quotient. *The logarithm of a quotient is found by subtracting the logarithm of the divisor from the logarithm of the dividend.* Thus if $P = M/N$, then $\log_{10} P = \log_{10} M - \log_{10} N$.

EXAMPLE. Divide 1512 by 36.4

$$\log_{10} 1512 = 3.17955$$
$$\log_{10} 36.4 = 1.56110$$

Then
$$\log_{10} (\text{Quot.}) = 1.61845$$
$$\text{Quot.} = 41.538$$

PROOF. If

$$M = 10^x, \quad \text{then} \quad x = \log_{10} M;$$

and if

$$N = 10^y, \quad \text{then} \quad y = \log_{10} N.$$

Dividing, we find

$$\frac{M}{N} = 10^{x-y}.$$

Therefore

$$\log_{10} \frac{M}{N} = x - y = \log_{10} M - \log_{10} N.$$

58. Logarithm of a Power. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

PROOF. If $N = 10^x$, then $x = \log_{10} N$, and $N^p = 10^{px}$. Hence

$$\log_{10} N^p = px = p \log_{10} N.$$

EXAMPLE. To find the square of 18 look up 180 in the column headed N. To the right of 180 in the column headed 0 is .25527 Thus

$$\log_{10} 18 = 1.25527$$

Since $(a)^2 = a^2$, we obtain the square of a number by making the exponent twice as large. Therefore

$$\log_{10} 18^2 = 2.51054$$

Looking for 51054 among the logarithms, we find it (very nearly) opposite 324, and in the column headed 0. Hence $18^2 = 324$.

59. Logarithm of a Root. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

EXAMPLE. To find the square root of 729, look for 729 in the column headed N. To the right of 729 in the column 0 is 86273. Thus

$$\log_{10} 729 = 2.86273$$

Since $\sqrt{a} = a^{1/2}$, we obtain the square root of a number by making the exponent $\frac{1}{2}$ as large. Therefore $\log_{10} \sqrt{729} = 1.43136$. Looking for .43136 among the logarithms, we find it opposite 270 in the column headed N (number), and in the column headed 0. Hence

$$\sqrt{729} = 27.0$$

PROOF. Let $N = 10^x$. Taking the rth root, we have $N^{1/r} = 10^{x/r}$. Hence

$$\log_{10} N^{1/r} = \frac{x}{r} = \frac{1}{r} x = \frac{1}{r} \log_{10} N.$$

The previous theorems show that the work in arithmetic processes can be greatly shortened. Thus addition is used in obtaining products; subtraction in obtaining quotients; multiplication in raising numbers to powers, and division in finding roots.

60. Interpolation. Numbers frequently contain more places than are given in a short table. It is then necessary to resort to what is called *interpolation*. Interpolation is based on the assumption that the change between two consecutive mantissas of the table is proportional to the change in the numbers.

EXAMPLE 1. Find the logarithm of 26.733
First write the characteristic, 1.

The mantissa for 2673 is .42700
The mantissa for 2674 is .42716
Their difference is 16 units in the last place.
The difference between 26730 and 26740 is 10.
The difference between 26730 and 26733 is 3.

Hence 3/10 of 16, which is 4.8 or 5, must be added to .42700; that is, $\log 26.733 = 1.42705$

NOTE. When the fraction of a unit in the part to be added is less than 0.5, it is neglected; when equal to or more than 0.5 the final figure is increased by one.

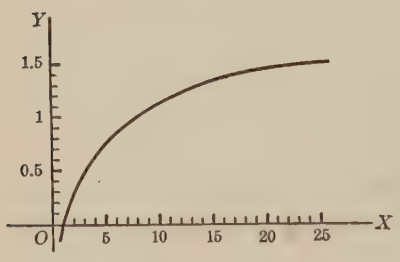
EXAMPLE 2. To find the number corresponding to a given logarithm. Given $\log_{10} N = 1.37658$, to find N.

The sequence of figures corresponding to the given mantissa is 238. Since the characteristic is 1, the number lies between 10 and 100, that is, there are two figures to the left of the decimal point; hence the required number is 23.8

EXAMPLE 3. Given $\log_{10} N = 8.49356 - 10$, required N.

The two consecutive mantissas between which .49356 lies are .49346 and .49360, whose difference is 14. The sequence of figures corresponding to the mantissa .49346 is 3115. The difference between 49346 and 49356 is 10. Therefore the fraction to be annexed to the sequence 3115 is 10/14 of 1, or .7; hence the required number is 0.031157

61. Graph of $y = \log_{10} x$. The values of x and y are plotted in the usual manner in rectangular coordinates. The graph is shown in Fig. 40. The unit on the y-axis is 10 times as large as the unit on the x-axis.



$y = \log x$

FIG. 40.

62. Accuracy. The accuracy required for nearly all engineering computations does not exceed 3 or 4 significant figures. Four-figure accuracy means that the errors permitted do not exceed a fraction

of 1 percent. Only a small portion of the fundamental data in science is reliable to this degree of accuracy. The usual measurements of the testing laboratory fall far short of it. In computing tax rates, or in solving certain problems in geodesy, compound interest, annuities, etc., a table with more decimal places is sometimes needed.

EXERCISES

1. Define: (*a*) exponent, (*b*) logarithm, (*c*) base.

2. Justify the following statements: (*a*) $a^{-n} = \frac{1}{a^n}$, (*b*) $a^n = \frac{1}{a^{-n}}$.

3. Explain why (*a*) $\log 1 = 0$, (*b*) $\log_{10} 10 = 1$.

4. From the graph, Fig. 40, find approximately the logarithms of the following numbers: 1, 3, 5, 8, 9, 10, 20.

5. Use logarithms to find the following indicated results:

(*a*) 17.42×9.625 (*d*) $8761 \div 23.45$ (*g*) $2150 \div 1728$. (*j*) 45^2.

(*b*) 8.431×23.78 (*e*) $1728 \div 231$. (*h*) $\sqrt{34.48}$ (*k*) $(1.06)^8$.

(*c*) 0.162×3449. (*f*) $2150 \div 231$. (*i*) $\sqrt[3]{965.2}$ (*l*) $(1.05)^{20}$.

6. How many cubic feet of water will a rectangular tank hold if it is 16.2 ft. long, 7.43 ft. wide, 8.46 ft. deep?

7. How many cubic feet of ensilage will a silo hold if it is 31.6 ft. high and the radius of the base is 7.4 ft.?

8. If interest is compounded annually at 5%, \$1 amounts to \$1.05 at the end of one year; $(1.05)^2$ at the end of two years; and $(1.05)^n$ in n years. Find the amount of \$465 at compound interest at 5% for 21 years.

NOTE. $A = \$465(1.05)^{21}$.

9. The average distance of the sun is 92 900 000 miles. If light travels 186 000 miles per second, how long will it take light to come from the sun to the earth? State your answer in minutes.

10. If light travels 186 000 miles per second, how long would it take light to travel from the sun to each of the planets? The distance in millions of miles from the sun to:

Mercury	is 36,	Jupiter	is 483.2,
Venus	" 67.2,	Saturn	" 886,
Earth	" 92.9,	Uranus	" 1782,
Mars	" 141.5,	Neptune	" 2792.

11. We are told in geography that one fourth of the earth's surface is land. How many square miles would this be if $r = 3960$ mi.?

12. In 1626 the Dutch bought Manhattan Island for \$24. To how much would this have amounted in 1926 if it had been placed at 6% compound interest?

13. What is the weight in tons of a ship whose "displacement" is 62 800 cubic feet?

14. In what time will a sum of money double itself at 8% compound interest? $\$200 = 100(1.08)^n$.

15. The average distance from the center of the earth to the center of the moon is 239 000 miles, and the moon revolves around the earth once in 27.32 days. Find the average velocity of the moon in (*a*) miles per hour, (*b*) feet per second.

16. The distance from the sun to the earth is 9.29×10^7 miles; to Neptune 2.79×10^9 miles; and to the nearest fixed star 2.5×10^{13} miles. If the distance from the sun to the earth is represented by 1 inch, what length would represent the distance from the sun to Neptune, using the same scale? Sun to nearest star?

17. The time in seconds of a complete oscillation of a pendulum is $T = 2\pi\sqrt{l/g}$ in which l is measured in feet and $g = 32.1$ Find T if (*a*) $l = 39$ inches, (*b*) $l = 24$ inches.

18. Assuming that the distance of the earth from the sun is 92 900 000 miles and that the earth describes a circle around the sun in $365\frac{1}{4}$ days, find the number of miles the earth moves in one minute. In one second.

19. It is estimated that the Mississippi River carries about 75×10^8 cubic feet of sediment to the gulf yearly. How thick a layer of sediment would this make if spread uniformly over one square mile? Spread over a county containing 600 square miles? Over the state of Iowa containing 56 000 square miles?

20. In each of the following exercises write the logarithm of the first member of the equation in terms of the logarithms of the quantities occurring in the second member.

(*a*) $A = \frac{1}{2}ab.$

(*b*) $C = 2\pi r.$

(*c*) $A = \pi r^2.$

(*d*) $V = \frac{4}{3}\pi r^3.$

(*e*) $A = \sqrt{s(s-a)(s-b)(s-c)}.$

(*f*) $A = \frac{h}{2}(a+b).$

(*g*) $S = 4\pi d^2.$

(*h*) $A = \frac{1}{2}ab \sin C.$

21. If a bullet is shot vertically upwards with an initial velocity of v, the height s to which it will rise is given by the equation $v = \sqrt{2gs}$. Find the minimum velocity at which a bullet can be fired upwards, if it is to strike an airplane 1685 feet high. Use $g = 32.1$

Find the approximate values of the following by the aid of logarithms.

22. $241.6 \times .00986$

23. 3.976×28.427

24. $.07964 \times 876.28$

25. $724 \times \sin 28°$.

26. $462 \times \cos 17°$.

27. $29.476 \times \tan 42°$.

28. $87.62 \div 27.394$

29. $962.4 \div 37.87$

30. $495.82 \div \sqrt{376.4}$

31. $29.876 \div \pi$.

32. $375.2 \div (-2.8476)$.

33. $\sqrt[3]{1896} \div 42.7$

34. $67\sqrt[3]{84.76}$

35. $(28/173)^{1/5}$.

36. $3/7 \times \sqrt[3]{84/579}$.

37. $\dfrac{26.7\sqrt{37.84}}{84.2\sqrt[3]{375.9}}$.

38. $\dfrac{162\sqrt[3]{84.72}}{42.7\sqrt[3]{87}}$.

39. The area A in acres of a triangular piece of ground whose sides are a, b, and c rods is given by the formula

$$A = \frac{\sqrt{s(s-a)\,(s-b)\,(s-c)}}{160},$$

where $s = (a + b + c)/2$. Compute the area, in acres, of the triangle whose sides are $a = 62$ rods, $b = 77$ rods, $c = 89$ rods.

CHAPTER IV

RADIAN MEASURE—TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

63. Magnitude and Measurement of Angles. In what precedes, and in elementary geometry, the angles considered were usually positive and less than two right angles. In more advanced mathematics, it is necessary to extend the idea so as to include angles of any magnitude. If a line, Fig. 41, continues to rotate so as to make more than one complete revolution, an angle greater than four right angles is generated. Such a line is said to generate an angle whose magnitude is determined by the amount and direction of the rotation. The magnitude may be given in degrees, radians, right angles, or revolutions. In elementary geometry the right angle is frequently used. The degree is generally used in practical problems involving numerical computation.

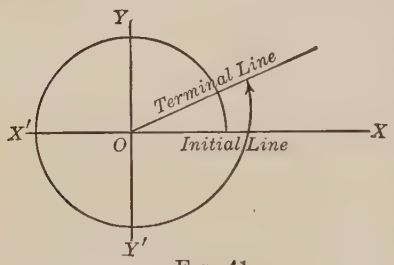


FIG. 41.

64. Positive and Negative Angles. As positive or negative directions may be assigned arbitrarily upon a line, so a positive or negative sense of generation may be assigned to an angle. A line is said to generate an angle if it first coincides with one side of the angle, then revolves about the vertex, and finally coincides with the other side. The rotating line is called the generating line of the angle.

Its first position is called the initial side of the angle and its final position the terminal side.

A positive angle is an angle generated by a line rotating in the positive direction, that is, opposite in direction to the hands of a clock. A negative angle is an angle generated by a line rotating in the negative direction, that is, clockwise. In Fig. 42, angle α is positive, and angle β is negative.

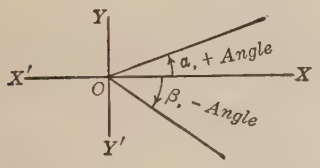


Fig. 42.

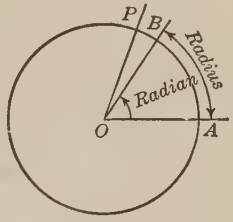


Fig. 43.

65. The Radian. The angle called a ***radian*** is the angle at the center of a circle subtended by an arc equal in length to the radius. Its value in terms of degrees is readily found. Since the circumference of a circle equals 2π times the radius, there are 2π radians in every revolution. Hence 2π radians equal 360°, or π radians = 180°.

$$\text{One radian} = \frac{180°}{\pi} = \frac{180°}{3.1416} = 57°.2958 = 57°\ 17'\ 45'',$$

or 57°.3, approximately. It follows, therefore, that the number of radians in an angle is equal to the number of degrees in the angle ÷ 57.2958

Note. 1 radian = 3437.7 minutes = 206265 seconds.

66. Radian Measure of Angles. The radian measure of an angle is, in general, the ratio of the length of the intercepted arc to the length of the radius. Expressed in the form of an equation, this reads:

Number of radians in an angle $= \dfrac{\text{length of intercepted arc}}{\text{length of the radius}}$.

Hence the *arc* = *radius* × *number of radians*. Denoting the radius by r, the angle in radians by θ, and the arc by s, we may write

$$s = r\theta, \quad \text{or} \quad \theta = \frac{s}{r}. \tag{1}$$

This relation is simpler than it would be if the angle were expressed in degrees.

NOTE. The name radian (which first appeared in 1873 in examination questions at Queen's College, Belfast, set by James Thomson, a brother of Lord Kelvin) is used to some extent in practical work, and universally used in various branches of higher mathematics.

67. The Use of π in Radian Measure. The number π is a numerical quantity, 3.1416 . . . , and not an angle. When we speak of 180° as π, or 90° as $\pi/2$, we always mean π radians, $\pi/2$ radians, etc. The student should become accustomed to expressing angles in circular measure, thus: 2π radians = 360°, $\pi/3$ radians = 60°, $\pi/6$ radians = 30°, etc.

68. Arc and Central Angle. Any angle can be measured by the number of radians it contains as well as the number of degrees. This greatly simplifies the study of circular motion. From geometry we have the theorem: In the same circle or equal circles, two central angles have the same ratio as their intercepted arcs. Hence, if the angle AOC (Fig. 44) is one radian, we have

$$\frac{\text{arc } AB}{\text{radius } (= \text{arc } AC)} = \frac{\text{angle } AOB}{\text{one radian } (= \text{angle } AOC)}$$

or arc AB = radius × angle AOB expressed in radians.

69. Applications. If an object is so distant that the central angle it subtends is very small, the length of the arc may be taken as the length of the object. The computation is then simplified by the use of radian measure.

EXAMPLE 1. If AB, Fig. 44, represents the height of a mountain whose peak has an angle of elevation of 1° 20′ of arc at a distance $OA = 30$ miles, find the height of the mountain above the point of observation.

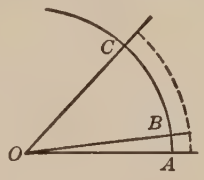


FIG. 44.

SOLUTION. Let $\angle AOC = 1$ radian, that is, arc $AC = OA = 30$; then

$$\frac{AB}{AC} = \frac{\angle AOB}{\angle AOC}, \quad \text{or} \quad \frac{AB}{30} = \frac{1° 20'}{57°.3} = \frac{80'}{3437'.7}.$$

and

$$AB = 30 \times 80 \div 3437.7 = 0.7 \text{ mi., nearly.}$$

EXAMPLE 2. Find the length of an arc which subtends an angle of 1° at the center of a circle whose radius is 20 ft.

SOLUTION. Since $s = r\theta$,

$$s = 20 \times \frac{1}{57.3} = \frac{20\pi}{180} = \frac{\pi}{9} = .349 \text{ ft.}$$

EXERCISES

1. Construct the following angles, using a protractor: 30°, 45°, 60°, 1 radian, π radians, $\pi/6$ radians, 400°, 760°.

2. How many degrees are there in a radian? 2 radians? 5 radians? 10 radians? 4π radians? 2π radians?

3. Through how many radians does the earth turn (rotate) in 24 hours? 6 hours? 3 hours?

4. Show by elementary geometry that a radian is less than 60°.

5. In a circle whose radius is 6 ft., the length of an intercepted arc is 15 ft. Find the angle in (*a*) radians, (*b*) degrees.

6. Find the distance in miles between two points on the earth's surface whose angular distance from each other, measured from the center of the earth, is 1 radian; 0.25 radian; 1 degree.

7. A mountain 10 miles away has an angle of elevation of 0.1 radian. About how high is its summit above the observer?

8. A mountain rising 4800 feet above an observer has an angle of elevation of 0.2 radian. Approximately how far away is it?

9. Find the length of arc which at a distance of 1 mile will subtend an angle of 1° at the eye. An angle of 10′. An angle of 1″.

10. A circle $2\frac{1}{2}$ ft. in radius is to be graduated, the graduations on

the circumference being 5 minutes of arc apart. Find in inches, correct to four decimal places, the length of arc between the graduations.

11. A large wheel on a dynamo has a velocity of 24 radians per second. If the wheel has a diameter of 16 feet, find the linear velocity of a point on the rim of the wheel.

12. A grindstone of Huron sandstone will safely stand a surface speed of 3750 feet per minute. How many revolutions per minute may a 30-inch stone be run? A one-foot stone? A two-foot stone?

13. Find the distance, measured on the surface of the earth, from the equator to a city in 41° north latitude. To a city 60° north. To Cedar Falls, Iowa, 42° 30′ north latitude. Use 3960 miles as the radius of the earth.

14. A pendulum 39 inches long vibrates 3° 15′ on each side of its mean position. Find the length of arc through which it swings.

15. Show that a ball, when viewed from a distance equal to 57 times its diameter, will subtend at the eye an angle of about 1°; at a distance of 3440 times the diameter the angle will be nearly 1′; at a distance of 206 000 times its diameter the angle will be almost exactly 1″.

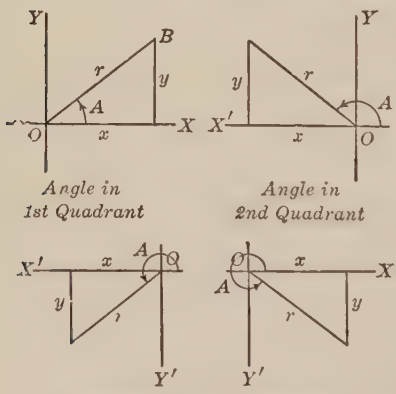


Fig. 45.

16. It is desired to construct a globe of such radius that the distance between two places on the same meridian, whose latitudes differ by 4½°, shall be 1 inch. Find the radius.

70. Trigonometric Functions of Any Angle Defined. Thus far trigonometric functions of an acute angle only have been considered. In solving oblique triangles, sometimes angles greater than 90° occur. We shall now consider a new set of definitions which apply to any angle whatever and which agree with the definitions given for acute angles.

If we regard the angle as formed by an initial line OX, and a line drawn from the origin to a point whose abscissa and ordinate are considered, the functions may be defined as follows:

$$\sin A = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}. \qquad \operatorname{ctn} A = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}.$$

$$\cos A = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}. \qquad \sec A = \frac{r}{x} = \frac{\text{distance}}{\text{abscissa}}.$$

$$\tan A = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}. \qquad \csc A = \frac{r}{y} = \frac{\text{distance}}{\text{ordinate}}.$$

The student should note that the abscissa and the ordinate are sometimes positive and sometimes negative, as explained in § 13, but the distance is always taken as positive.

The relations given in § 34 remain true, and the definitions given above apply for angles of any size and include the definitions of § 33 as a special case. The student should review them.

71. Trigonometric Functions as Lines. Since the trigonometric functions of any angle are ratios, they may be represented by lines if we first choose a unit of length, and then construct right triangles such that the denominators of the ratios are equal to this unit.

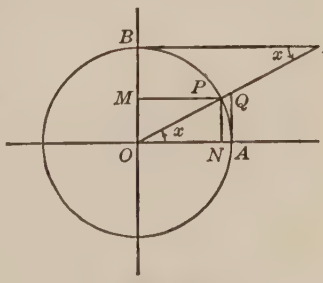


Fig. 46.

If, in Fig. 46, the radius is taken equal to 1, the circle is called a unit circle. Then $OA = OP = OB = 1$. Constructing perpendiculars as shown in the figure, we have

$$\sin x = \frac{NP}{OP} = NP, \qquad \operatorname{ctn} x = \frac{BR}{OB} = BR,$$

$$\cos x = \frac{ON}{OP} = ON, \qquad \sec x = \frac{OQ}{OA} = OQ,$$

$$\tan x = \frac{AQ}{OA} = AQ, \qquad \csc x = \frac{OR}{OB} = OR.$$

In each case the denominator of the fraction is 1.

72. Reduction of Trigonometric Functions to Functions of Acute Angles. In Fig. 47, let YY' be drawn perpendicular to the line XX', and let the diameters PR and SQ be drawn so as to make angle MOP equal to angle NOQ. Then the distances OP, OQ, OR, and OS are equal. From P, Q, R and S draw perpendiculars to the x-axis. The triangles MOP, NOQ, NOR, and MOS are congruent, having the hypotenuse and an acute angle of one equal, respectively, to the hypotenuse and an acute angle of the other. Considering only the absolute values of the functions, sin XOP, sin XOQ, sin XOR, sin XOS, all have the same numerical value; and so on for each of the other functions. Hence for every acute angle in the first quadrant there is an angle in each of the other quadrants whose functions, in absolute value, are equal to the same function of this acute angle.

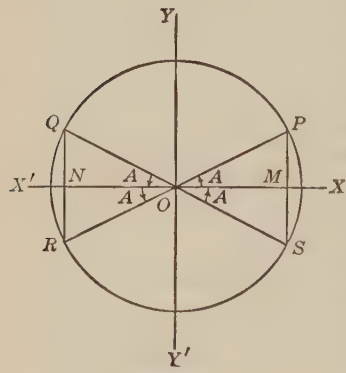


Fig. 47.

If we let $\angle XOP = A$, and prefix the proper signs to the functions of the angles XOQ, $(180° - A)$; XOR, $(180° + A)$; and XOS, $(360° - A)$; or XOS, $(-A)$; remembering that ON, NR, and MS are negative, we have:

ANGLE IN QUADRANT II, (XOQ).

$$\sin (180° - A) = \frac{NQ}{OQ} = \sin A.$$

$$\cos (180° - A) = \frac{ON}{OQ} = -\cos A.$$

$$\tan (180° - A) = \frac{NQ}{ON} = -\tan A.$$

ANGLE IN QUADRANT III, (XOR).

$$\sin (180° + A) = \frac{NR}{OR} = -\sin A.$$

$$\cos (180° + A) = \frac{ON}{OR} = -\cos A.$$

$$\tan (180° + A) = \frac{NR}{ON} = \tan A.$$

ANGLE IN QUADRANT IV, (XOS).

$$\sin (360° - A) = \frac{MS}{OS} = -\sin A.$$

$$\cos (360° - A) = \frac{OM}{OS} = \cos A.$$

$$\tan (360° - A) = \frac{MS}{OM} = -\tan A.$$

If, in any such case, A is greater than 45°, we would have to apply § 41, p. 35, if we wish to make use of the tables. Thus, if $A = 90° - B$, we have, by § 41, $\sin A = \cos B$, etc. But we have also

$$(180° - A) = (180° - (90° - B)) = 90° + B,$$

where B is less than 45° if A is greater than 45°. Hence we may rewrite the formulas for Quadrant II as follows:

$$\sin (90° + B) = \cos B,$$
$$\cos (90° + B) = -\sin B,$$
$$\tan (90° + B) = -\operatorname{ctn} B.$$

Similar formulas may be written in each quadrant; these are convenient when A exceeds 45°.

73. Functions of a Negative Angle, $-A$. Since A and $-A$ are numerically equal, it is evident that if one is in the first quadrant, the other is in the fourth; if one is in the second quadrant, the other is in the third. Therefore, for all values of A and $-A$, the abscissas have the same sign and the ordinates have opposite signs. Hence

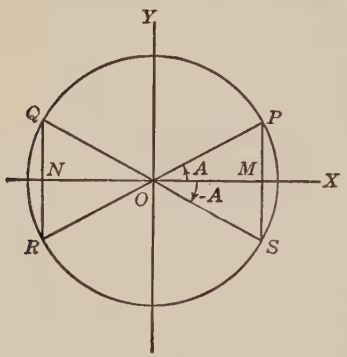


Fig. 48.

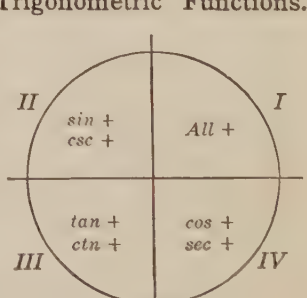
$$\sin(-A) = -\sin A.$$
$$\cos(-A) = \cos A.$$
$$\tan(-A) = -\tan A.$$

Theorem. *The functions of any angle, however great, can be reduced to the functions of an acute angle.*

For example,

$$\sin 152° = \sin(180° - 28°) = \sin 28°,$$
$$\sin 205° = \sin(180° + 25°) = -\sin 25°,$$
$$\sin 353° = \sin(360° - 7°) = -\sin 7°,$$
$$\sin(-10°) = -\sin 10°.$$

74. Algebraic Signs of the Trigonometric Functions. Keeping in mind the algebraic signs of abscissas and ordinates in the different quadrants, and remembering that the distance is always taken as positive, we have at once from the definitions of the trigonometric functions:



Rule for Signs
Fig. 49.

In quadrant I, all functions are positive.

In quadrant II, sin and csc only are positive.

In quadrant III, tan and ctn only are positive.

In quadrant IV, cos and sec only are positive.

These results are exhibited in Fig. 49. All functions not indicated in each quadrant are negative.

If an angle is greater than 360°, its functions are evidently equal to the same functions of the angle less than 360° which is found by subtracting 360° (once or more) from the given angle.

EXERCISES

1. Represent by lines on the unit circle the functions of

(*a*) 30°. (*b*) 45°. (*c*) 60°. (*d*) 85°. (*e*) 15°.

2. From Fig. 46, when x is an acute angle, determine which is the greater:

(*a*) $\sin x$ or $\tan x$. (*c*) $\csc x$ or $\operatorname{ctn} x$. (*e*) $\cos x$ or $\operatorname{ctn} x$.
(*b*) $\sin x$ or $\sec x$. (*d*) $\sec x$ or $\tan x$. (*f*) $\cos x$ or $\csc x$.

3. In each of the following exercises, construct the triangle, compute the remaining side, and read off all the functions of angle A.

(*a*) $\sin A = \frac{3}{5}$. (*d*) $\tan A = y$. (*g*) $\sin A = y$.
(*b*) $\cos A = \frac{1}{3}$. (*e*) $\sin A = 1/\sqrt{3}$. (*h*) $\cos A = x$.
(*c*) $\tan A = 3$. (*f*) $\cos A = \frac{1}{2}$. (*i*) $\sec A = \sqrt{5}$.

4. A dynamo makes 60 revolutions per second. Through how many degrees does it turn per second? How many radians? Through how many right angles?

5. Through what angle does the earth rotate in 16 hours? 18 hours? 30 hours? 36 hours?

6. Find the complements of the following angles:

30°, 60°, 45°, 20°, 65°, 89°, 2°, 7°, 10°, 72°, 40°, 50°, x°.

7. Express the following as functions of the complementary angle:

(*a*) $\sin 70°$. (*d*) $\cos \frac{\pi}{3}$. (*g*) $\cos 12°$.
(*b*) $\cos 84°$. (*e*) $\sec 81°$. (*h*) $\tan 6° 15'$.
(*c*) $\sin \frac{\pi}{3}$. (*f*) $\csc \frac{\pi}{6}$. (*i*) $\operatorname{ctn} 65° 40'$.

8. Find the supplements of the following angles: 150°, 170°, 135°, 120°, 100°, 175°, 178°, 95°.

9. Express the following as functions of an acute angle:

(*a*) $\sin 120°$. (*c*) $\tan 120°$. (*e*) $\cos 155°$.
(*b*) $\cos 120°$. (*d*) $\sin 175°$. (*f*) $\tan 178°$.

10. Express the following as functions of an acute angle:

(*a*) sin 210°. (*c*) tan 210°. (*e*) cos 230°.
(*b*) cos 210°. (*d*) sin 200°. (*f*) tan 260°.

11. Express the following as functions of an acute angle:

(*a*) sin 300°. (*b*) cos 330°. (*c*) tan 320°.

12. Express as functions of a positive acute angle:

(*a*) sin (−45°). (*b*) cos (−20°). (*c*) tan (−50°).

13. In what quadrant does an angle lie if:

(*a*) its sine is positive and its cosine is negative.
(*b*) its tangent is negative and its cosine is positive.
(*c*) its sine is negative and its cosine is positive.
(*d*) its cosine is positive and its tangent is positive.
(*e*) its sine is negative and its cosine is negative.
(*f*) its cosine is negative and its tangent is positive.

14. Find the other trigonometric functions of an angle x:

(*a*) Given $\sin x = \frac{1}{2}$, and x in the second quadrant.
(*b*) Given $\cos x = -\frac{1}{2}$, and x in the third quadrant.
(*c*) Given $\tan x = -\sqrt{3}$, and x in the fourth quadrant.

15. (*a*) Given $\sin x = -\frac{2}{3}$, and x in the third quadrant.
(*b*) Given $\cos x = \frac{3}{5}$, and x in the fourth quadrant.
(*c*) Given $\tan x = -\frac{3}{7}$, and x in the second quadrant.
(*d*) Given $\sin x = -\frac{1}{2}$, and $\tan x$ positive.
(*e*) Given $\cos x = \frac{3}{4}$, and $\tan x$ negative.

16. Express as functions of a positive acute angle:

(*a*) sin 305°. (*d*) sin (−57°). (*g*) cos (−330°).
(*b*) cos (−38°). (*e*) ctn 341°. (*h*) tan (−185°).
(*c*) tan (−2°). (*f*) sin (−152°). (*i*) sin 729°.

17. Find the values of the following:

(*a*) cos 510°. (*d*) sec 220°. (*g*) ctn 170°.
(*b*) tan (−48°). (*e*) csc 315°. (*h*) tan 147°.
(*c*) sin (−160°). (*f*) cos (−22°). (*i*) cos 870°.

18. Solve the following equations, giving all positive values of x less than 360°.

(*a*) $\sin x = \pm 1$. (*f*) $\sec^2 x = \frac{4}{3}$. (*k*) $\sin^2 x = \frac{1}{3}$.
(*b*) $\cos x = \pm 1$. (*g*) $\tan^2 x = 3$. (*l*) $\csc^2 x = 1$.
(*c*) $\tan^2 x = 1$. (*h*) $\sin^2 x = \frac{3}{4}$. (*m*) $\sin^2 x = .09$
(*d*) $\cos^2 x = \frac{1}{4}$. (*i*) $\cos^2 x = \frac{1}{2}$. (*n*) $\tan^2 x = 2$.
(*e*) $\sin^2 x = \frac{1}{4}$. (*j*) $\tan x = \pm 2$. (*o*) $\sin x = .49$

75. Functions of 0°, 90°, 180°, 270°, 360°. In the unit circle let angle AOB be any angle and BC be the perpendicular on $A'A$. As angle AOB approaches 0°, 180°, or 360°, $OB = 1$ and BC approaches zero as a limit. As angle AOB approaches 0° or 360°, OC approaches 1; as angle AOB approaches 180°, OC approaches -1. At these limiting positions, we may write:

$\sin 0° = 0/1 = 0$, $\sin 180° = 0/1 = 0$, $\sin 360° = 0/1 = 0$,

$\cos 0° = 1/1 = 1$, $\cos 180° = -1/1 = -1$, $\cos 360° = 1/1 = 1$,

$\tan 0° = 0/1 = 0$, $\tan 180° = 0/-1 = 0$, $\tan 360° = 0/1 = 0$.

If angle AOB approaches 90° or 270°, $OB = 1$, BC approaches 1 at 90° and -1 at 270°, as a limit; OC approaches 0 as a limit; and $OA = 1$. AB' increases without limit as angle AOB approaches 90° or 270°. Hence we say

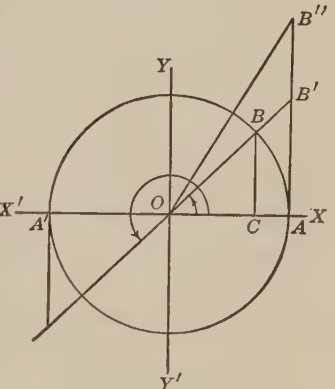


Fig. 50.

$\sin 90° = 1/1 = 1$,

$\sin 270° = -1/1 = -1$,

$\cos 90° = 0/1 = 0$,

$\cos 270° = 0/1 = 0$,

$\tan 90° = \infty$, $\tan 270° = -\infty$.

Note. The symbol ∞, read "infinity," means without limit; if a variable y becomes and remains larger than any positive number k, however great, we say that y increases without limit, or that y becomes infinite. Infinity is not a number in the sense in which 3, -200, $\sqrt{5}$ are numbers. Hence also tan 90° does not exist in the sense of an ordinary number. It may be regarded as a manner of speech rather than a number that can be added, subtracted, multiplied or divided as ordinary numbers are.

Summarizing these results, we have the following table:

Function	0°	90°	180°	270°	360°
Sine	0	1	0	−1	0
Cosine	1	0	−1	0	1
Tangent	0	∞	0	−∞	0

Sines and cosines vary between 1 and − 1, tangents between − ∞ and ∞.

76. Solution of Oblique Triangles. One of the most useful applications of mathematics is the systematic location and relocation of points and lines on the earth's surface and the measurement of portions of that surface—the art of surveying. In surveying an extensive region, two points several miles apart are selected for the extremities of a so-called base-line. The length of such a line is measured with great accuracy and may be used in determining the length of a degree on a given parallel or meridian, the length of the earth's diameter, the distance across a river, the distance to the moon or sun. The base-line used in measuring the distance to the nearer stars is the diameter of the earth's orbit. The student will find numerous applications for the methods of solving oblique triangles in surveying, engineering, physics, astronomy, and navigation.

The principle underlying the formulas used in such work is that the corresponding sides of similar triangles are proportional. In this chapter a method for solving any triangle will be developed. The six parts of a triangle are the three sides and the three angles. For a solution to be possible three parts must be known, at least one of which must be a side. This gives rise to four cases, namely,

1. *Two sides and the included angle.*
2. *Three sides.*
3. *One side and two angles.*
4. *Two sides and an angle opposite one of them.*

Since there are six parts to a triangle and in each of the four cases three parts are given, three independent formulas or relations connecting these parts are necessary for a complete solution. These relations are:

1. The sum of the angles of a triangle is equal to 180°.
2. The sine theorem, or the *law of sines.*
3. The cosine theorem, or the *cosine law.*

For the convenience of carrying out numerical computations various other relations are derived from the formulas growing out of the sine theorem and cosine theorem.

77. Area of an Oblique Triangle.

1. *When two sides and the included angle are known.*

In the triangle ABC denote the area by S, and the altitude on AB by h. Then

$$S = \tfrac{1}{2}ch. \tag{2}$$

Now

$$h = b \sin A. \tag{3}$$

Substituting, we find

$$S = \tfrac{1}{2}bc \sin A. \tag{4}$$

Similarly,

$$S = \tfrac{1}{2}ac \sin B, \text{ or } \tfrac{1}{2}ab \sin C. \tag{5}$$

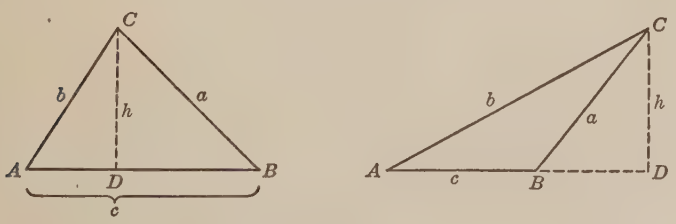


FIG. 51.

STATEMENT. *The area of a triangle equals one-half the product of any two sides multiplied by the sine of the angle included by these sides.*

EXAMPLE 1. Find the area of the oblique triangle in which $b = 56$, $c = 59$, and $A = 39°$.

SOLUTION BY THE USE OF NATURAL FUNCTIONS.

$$\begin{aligned}\text{Area} &= \tfrac{1}{2} \times 56 \times 59 \times \sin 39°.\\ &= 28 \times 59 \times \sin 39°.\\ &= 28 \times 59\ (.6293).\\ &= 1039.6 \text{ square units.}\end{aligned}$$

EXAMPLE 2. Find the area of the oblique triangle in which $b = 97$, $c = 84$, and $A = 43° 30'$.

SOLUTION BY THE USE OF LOGARITHMS.

$$\begin{aligned}A &= \tfrac{1}{2} \times 84 \times 97 \times \sin 43° 30'\\ &= 42 \times 97 \times \sin 43° 30'\\ \log 42 &= 1.62325\\ \log 97 &= 1.98677\\ \log \sin 43° 30' &= 9.83781 - 10\\ \log A &= 3.44783\\ A &= 2804.3 \text{ square units.}\end{aligned}$$

2. *When three sides are known.*

From plane geometry it is known that when three sides a, b, and c are given the area of a triangle is found by using the formula:

$$S = \sqrt{s(s-a)(s-b)(s-c)}, \tag{6}$$

where $s = \frac{1}{2}(a + b + c)$.

It follows also, from geometry, that the radius r of the circle inscribed in the triangle is

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{7}$$

NOTE. Formula (6) was given by Heron of Alexandria in the first century A.D. and is called Heron's formula. Heron was a teacher of mathematics and mechanics at Alexandria in Egypt. An extension of the formula was given by Brahmagupta, a Hindu mathematician of about 628 A.D. His formula gives the area of any quadrilateral in-

scribed in a circle:

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

The formula for the area of a triangle may be regarded as the special case where $d = 0$.

EXAMPLE 1. Find the area of the triangle whose sides are $a = 39$ rods, $b = 42$ rods, $c = 45$ rods.

SOLUTION.

$$\begin{aligned} a &= 39 \\ b &= 42 \\ c &= 45 \\ \hline a+b+c &= 126 \end{aligned}$$

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

$$s = \frac{a+b+c}{2} = 63.$$

Substituting, we have

$$\begin{aligned} s-a &= 24 \\ s-b &= 21 \\ s-c &= 18 \end{aligned}$$

$$\begin{aligned} S &= \sqrt{63 \times 24 \times 21 \times 18} \\ &= 756 \text{ sq. rods} \\ &= 4.725 \text{ acres,} \end{aligned}$$

since one acre = 160 sq. rods.

EXAMPLE 2. Find the area of the triangle whose sides are $a = 82$ rods, $b = 79$ rods, $c = 37$ rods.

SOLUTION. $s = 99$, $s - a = 17$, $s - b = 20$, $s - c = 62$. Hence

$$S = \sqrt{99 \times 17 \times 20 \times 62}$$

$$\begin{aligned} \log 99 &= 1.99564 \\ \log 17 &= 1.23045 \\ \log 20 &= 1.30103 \\ \log 62 &= 1.79239 \\ & \overline{6.31951} \\ \log \text{Area} &= 3.15976 \\ \text{Area} &= 1444.6 \text{ sq. rods,} \\ &= 9.03 \text{ acres nearly.} \end{aligned}$$

EXERCISES

1. Construct a triangle, having given:
   - (*a*) Two sides and the included angle.
   - (*b*) Three sides.
   - (*c*) One side and two angles.
   - (*d*) Two sides and the angle opposite one of them.
2. Prove that the sum of the angles of a triangle equals 180°.

3. Find the area of the triangle lying between the coordinate axes and the line $2x + 5y = 23$.

4. Find the area of the triangle bounded by the x-axis and the lines: $y = 2x$ and $x = 17$.

5. Find the area of the triangular field, the coordinates of whose vertices are (0, 0) (12, 2) and (5, 7). What kind of triangle is it?

6. Find the area of the triangle formed by the lines:

(*a*) $x - 5y = -4$, (*b*) $5x + y = 58$, (*c*) $37x - 3y = 34$.

What kind of triangle is it?

7. Find the area of the following triangles, having given:

(*a*) $b = 30$,	$c = 40$,	$A = 28°$.
(*b*) $b = 96$,	$c = 50$,	$A = 41°$.
(*c*) $a = 42$,	$b = 38$,	$C = 22°$.
(*d*) $a = 65$ rds.,	$c = 56$ rds.,	$B = 47°$.
(*e*) $a = 42$ rds.,	$c = 56.2$ rds.,	$B = 39°$.

8. Find the areas of the following triangles, having given:

(*a*) $a = 37$ rds.,	$b = 35$ rds.,	$c = 12$ rds.
(*b*) $a = 96$ rds.,	$b = 84$ rds.,	$c = 102$ rds.
(*c*) $a = 24$ rds.,	$b = 31$ rds.,	$c = 42$ rds.
(*d*) $a = 296$ ft.,	$b = 314$ ft.,	$c = 311$ ft.
(*e*) $a = 541$ ft.,	$b = 762$ ft.,	$c = 649$ ft.

NOTE. The area A in acres of a triangular piece of ground, whose sides are a, b, and c rods, is given by the formula:

$$A = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{160}.$$

9. Compute the areas, in acres, of the following triangles:

(*a*) $a = 72$ rds.,	$b = 84$ rds.,	$c = 92$ rds.
(*b*) $a = 16$ rds.,	$b = 75$ rds.,	$c = 81$ rds.
(*c*) $a = 21$ rds.,	$b = 33$ rds.,	$c = 41$ rds.

10. The sewer assessment of a certain city improvement district is 2.254 cents per square foot. What is the total assessment on a triangular lot whose sides measure 194 ft., 222 ft., and 116 ft., respectively?

11. A surveyor finds that the sides of a triangular field measure: $AB = 96.2$ rds., $BC = 87.9$ rds., $CA = 76.4$ rds. Find the number of acres in the field.

12. A boy wishes to determine accurately the number of acres in a garden plot $ABCD$, in the shape of a quadrilateral. The measurements are $AB = 15.1$ rds., $BC = 14.3$ rds., $CD = 8.6$ rds., $DA = 17.8$ rds., $AC = 21.4$ rds. He computes the area to be 182.4 square rods. Is this result correct? If the result is not correct, find how much his error is.

13. A railroad cuts off from a man's land a triangular piece of land ABC. Side $AB = 21.1$ rds., $AC = 19.4$ rds., and angle $A = 37° 30'$. A neighbor offers \$100 for the piece of land. How much is this per acre?

14. A store completely covers a triangular lot which extends 70 ft. and 85 ft., respectively, on the two sides from a corner. The angle at the corner between the streets is 62°. Find the number of square feet in the lot.

15. A corner lot measures 137.2 ft. and 166.6 ft., respectively, from the corner along the two streets which intersect at an angle of 72° 50′. What will be the amount of the assessment on this lot at the rate of .894¢ per square foot?

78. Law of Sines. A theorem of plane geometry states that if two sides of a triangle are unequal, the angles opposite them are unequal in the same order. The converse is also true. The exact relation between the sides is given by the following theorem.

Law of Sines. *In any triangle any two sides are proportional to the sines of the angles opposite them.*

Let ABC be any plane triangle. Draw the perpendicular h from the vertex C of the triangle to the opposite side AB (Fig. 52), or AB produced (Fig. 53).

In the right triangle ADC, Fig. 52,

$$h = b \sin A,$$

and in the right triangle BDC,

$$h = a \sin B.$$

In the right triangle ADC, Fig. 53,

$$h = b \sin A,$$

and in the right triangle BDC,

$$h = a \sin (180° - B)$$
$$= a \sin B.$$

Hence, whether the triangle is acute or obtuse, we have

$$h = a \sin B = b \sin A.$$

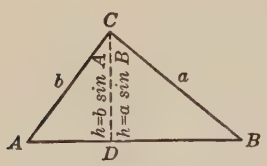


Fig. 52.

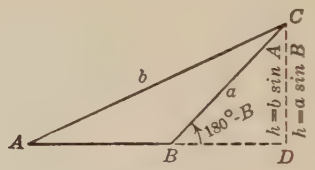


Fig. 53.

Dividing by sin A sin B, we find

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \tag{8}$$

Similarly, by drawing a perpendicular from A to BC, or BC extended, we may obtain

$$\frac{b}{\sin B} = \frac{c}{\sin C}. \tag{9}$$

Equating the values in (8) and (9), we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \tag{10}$$

which is known as the *law of sines.*

The above law may be used to solve a triangle, (1) when two angles and a side are given, (2) when two sides and an angle opposite one of them are given.

Example. Solve the triangle, having given $A = 41°$, $B = 56°$, $a = 65$.

Solution. Since two angles A and B are given, the third angle is obtained by subtracting their sum from 180°. Therefore,

$$C = 180° - (A + B) = 180° - 97° = 83°.$$

Since the side opposite A is known, the law of sines may be used.

$$\frac{b}{\sin 56°} = \frac{65}{\sin 41°}, \qquad \text{whence} \qquad b = \frac{65 \sin 56°}{\sin 41°}.$$

Substituting the numerical values of sin 56°, and sin 41° from the table, there results:

$$b = \frac{65 \times .829}{.6561} = 82.1$$

Similarly, to find side c,

$$\frac{c}{\sin 83°} = \frac{65}{\sin 41°}.$$

Solving for c, we find

$$c = \frac{65 \sin 83°}{\sin 41°} = 98.3$$

Since we know all the sides and all the angles, the triangle is said to be solved.

79. The Ambiguous Case. To solve a triangle when two sides and an angle opposite one of them are given, we first find the unknown angle which lies opposite one of the given sides. Since this angle is determined from its sine it admits of two values (C or $180° - C$), which are supplements of each other; hence either value of this angle may be taken, unless one of them is excluded by the conditions of the problem.

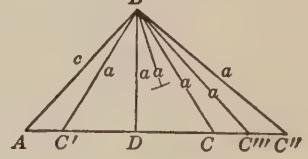


FIG. 54.

Let sides a and c, and the angle A be the parts given.

(*a*) If $a < BD$, BD being perpendicular to AC, there is no solution.

(*b*) If $a = BD$, the triangle is a right triangle and there is one solution, ADB, unless the given angle A is obtuse, in which case there is no solution.

(*c*) If $a = c$, then by geometry $A = C$, both A and C are acute, and the triangle is isosceles, $AC'''B$.

(*d*) If $a > c$, then by geometry $A > C$, and C must be acute. Hence there is only one solution, namely, ABC''.

(*e*) If $a < c$ but greater than BD, $A < C$, and A must be acute in order that the triangle may be possible. If A is an acute angle, either of two triangles ABC' or ABC will fulfill the conditions. But angles $AC'B$ and ACB are supplementary. Hence in using the law of sines,

$$\sin C = \frac{c \sin A}{a},$$

we get the value of the acute angle C from the table, and as a second solution the supplementary obtuse angle:

$$C' = 180° - C.$$

EXAMPLE. Determine the number of solutions when A, c, and a are given, A being 30° and $c = 10$.

SOLUTION. The length of the perpendicular $BD = 10 \sin 30° = 5$. Hence if $a < 5$, there is no solution; if $a = 5$, one solution, a right triangle; if $a > 5$, but $a < 10$, there are two solutions, namely, $AC'B$ and ACB; $AC'B = 180° - ACB$. If a is equal to or greater than c, there is one solution in each case.

80. Historical Note. The sine law was discovered by an Arabian (Persian by birth) mathematician, Nasir ed-Din, 1250 A.D. To him we owe the first systematic treatise on plane trigonometry. He used a combination of Greek trigonometry, employing chords, with the Hindu trigonometry employing sines. The law of sines was introduced into Europe by Regiomontanus, in a work entitled *De Triangulis*, published posthumously in 1533.

EXERCISES

1. Having given $a = 10$, $C = 88°$, $A = 60°$; solve the triangle.

2. Having given $a = 20$, $B = 75°$, $A = 36°$; solve the triangle.

3. Two sides of a triangle are 40 and 54 and the angle opposite the latter is 65°. Find the other parts.

4. Two sides of a triangular lot are 210 feet and 175 feet and the

angle opposite the former is 72°. Find the length of a fence around the lot.

5. Telescopes, at the end of a base-line 250 feet long on the deck of a ship, are turned upon a distant fort. The lines of sight of the telescope are found to make angles of 89° 12′ and 89° 40′, respectively, with the base-line. Find the distance from the fort to the nearer telescope. What theorem in geometry do you use in determining the side to be computed?

6. Two observers stationed on opposite sides of a cloud and in the same vertical plane with it observe its angles of elevation to be 44° 20′ and 62° 30′, respectively. Their distance from each other is 1800 feet. What is the distance of the cloud from each observer?

7. Two straight highways meet at A and make an angle of 60° with each other. At B, 6 miles from A and on one of the highways, a cross road 5.5 miles long runs to C, situated on the other highway. How far is C from A? Ans. 1.2 mi. or 4.8 mi.

8. A "shooting star" is observed from two stations 3 miles apart on the earth's surface; the angle of elevation of the star at one station is 88° 10′; and at the other is 87° 30′, the star being in the same vertical plane with the two stations. Making no allowance for the curvature of the earth, find the height of the star.

9. A lighthouse was observed from a ship to bear N. 36° E.; after the ship sailed due south 4 miles, the lighthouse bore N. 20° E. Find the distance of the lighthouse from the ship in each position.

10. An observer finds the angle of elevation of a sharply defined cloud, which is in the same vertical plane with the sun, to be 46°. The altitude of the sun at the same time measures 51° 30′. If the shadow of the cloud is 810 feet from the observer, find the height of the cloud.

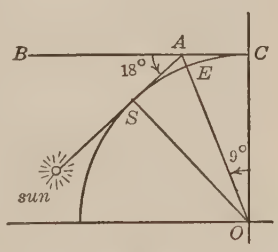


Fig. 55.

11. When the sky is clear, twilight ceases when the sun has reached a position 18° below the horizon, $\angle BAS = 18°$. Find the height AE of the atmosphere which is sufficiently dense to reflect the sun's rays Take $OC = 3960$ miles. Diminish the result by 20% because of refraction.

12. It is required to find the horizontal distance from a point A to an inaccessible point B on the opposite bank of a river. A point C is

located on the same side of the river as A and the following measurements taken:

$AC = 320$ feet, angle $CAB = 58°$, angle $ACB = 67° 39'$. Find AB.

13. To find the distance from a point A to the point B at the top of a railroad embankment, a point C, lying at the foot of the embankment and in the same vertical plane as A and B, was selected. If angle $BAC = 36°$, $AC = 42$ feet, and $BC = 54$ feet, find AB.

14. To determine the distance of a hostile fort A from a place B, a line BC and the angles ABC and BCA were measured and found to be 540 feet, 76° and 82°, respectively. Find the distance AB.

15. To find the distance between two mountain peaks A and B, a line is run from B to C, a distance of 5 miles. The angle $ABC = 70°$ and the angle ACB 84°. Find AB.

16. Two observers are 3 miles apart on a level plain, and facing each other. They find that the angles of elevation of a balloon which is in the same vertical plane with themselves are 52° and 81°, respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

17. A balloonist is directly above a straight road 5 miles long, joining two towns. He observes that the direction of one town makes an angle of 58° and the other an angle of 66° with the perpendicular. Find the distance of the balloon from each town and also the height of the balloon.

81. Law of Cosines. *In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these two sides by the cosine of their included angle.*

Proof. Let it be required to find a, having given the sides b and c and the angle A.

Case I. When angle A is acute, it is proven in geometry that: The square of the side opposite an acute angle equals the sum of the squares of the other two sides minus twice the product of one of these sides into the projection of the other upon it. Applying this theorem to our triangle, we have:

$$CB^2 = AC^2 + AB^2 - 2AB \times AD,$$

or

$$a^2 = b^2 + c^2 - 2c \times AD. \tag{11}$$

But

$$AD = b \cos A.$$

Substituting in (11),

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{12}$$

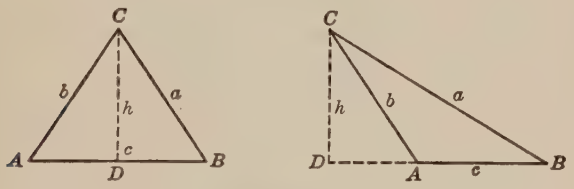


Fig. 56.

Case II. If angle A is obtuse, the following theorem is proven in elementary geometry:

The square of the side opposite an obtuse angle equals the sum of the squares of the other sides plus twice the product of one of these sides into the projection of the other upon it.

Applying this theorem to our triangle, we have:

$$CB^2 = AC^2 + AB^2 + 2AB \times AD,$$

or

$$a^2 = b^2 + c^2 + 2c \times AD. \tag{13}$$

But

$$\begin{aligned} AD &= b \cos DAC \\ &= b \cos (180^\circ - A) \\ &= -b \cos A. \end{aligned}$$

Hence in any case

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{14}$$

Similarly we may find

$$b^2 = a^2 + c^2 - 2ac \cos B, \tag{15}$$

and

$$c^2 = a^2 + b^2 - 2ab \cos C. \tag{16}$$

If in (14) $A = 90°$, then $\cos A = 0$ and (14) becomes

$$a^2 = b^2 + c^2,$$

which is the known relation between the sides of a right triangle, where A is the right angle. Solving (14), (15), (16) for the cosines of the angles, we obtain

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \tag{17}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \tag{18}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}. \tag{19}$$

The above law is used to solve a triangle (1) when two sides and the included angle are given, or (2) when the three sides are given.

Note. The law of cosines is essentially a geometric theorem of Euclid. In its present form it was given by the French mathematician Vieta (1540–1603). Although he was a lawyer in the service of the state under Henry III and Henry IV, his fame rests entirely on his achievements in mathematics, which he studied as a diversion. So great was his devotion to this science that he would remain in his room for several days at a time, eating and sleeping merely enough to sustain himself.

Example 1. Having given $b = 12$, $c = 15$, and $A = 41°$, solve the triangle.

Solution. To find side a use the formula:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= 144 + 225 - 360 \cos 41° \\ &= 369 - 360\ (.7547) \\ &= 97.308 \end{aligned}$$

Therefore,

$$a = \sqrt{97.308} = 9.864$$

To find the angles B and C use the law of sines.

$$\sin C = \frac{c \sin A}{a} = \frac{15 \sin 41°}{9.864}; \quad \text{therefore,} \quad \angle C = 86° 3'.$$

$$\sin B = \frac{b \sin A}{a} = \frac{12 \sin 41°}{9.864}; \quad \text{therefore,} \quad \angle B = 52° 57'.$$

CHECK. $A + B + C = 41° + 52° 57' + 86° 3' = 180°$.

EXAMPLE 2. Having given $a = 8$, $b = 10$, $c = 12$, required to find the angles:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{100 + 144 - 64}{2 \times 10 \times 12}; \quad \text{therefore,} \quad A = 41° 25'.$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{64 + 144 - 100}{2 \times 8 \times 12}; \quad \text{therefore,} \quad B = 55° 46'.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 100 - 144}{2 \times 8 \times 10}; \quad \text{therefore,} \quad C = 82° 49'.$$

CHECK. $A + B + C = 180°$.

NOTE. The above formulas (14) to (19) inclusive are not adapted to logarithms. Computation by them is usually long unless the sides are numbers with only one or two significant digits. Formulas adapted to logarithmic computation will be given in a later chapter. Formula (14) is used frequently in the proof of theorems in pure and applied mathematics.

EXERCISES

1. Having given $a = 10$, $c = 8$, $B = 60°$; solve the triangle.
2. Having given $a = 15$, $b = 20$, $C = 60°$; solve the triangle.
3. Having given $a = 25$, $c = 30$, $B = 60°$; solve the triangle.
4. Having given $b = 20$, $c = 30$, $A = 60°$; solve the triangle.
5. In order to find the distance between two points A and B separated by a pond, a third point C was selected. The distance $CB = 102$ rds., $CA = 90$ rds., and angle $BCA = 43°$. Find the distance from A to B.
6. Under what visual angle is an object 7 ft. long seen, when the eye of the observer is 11 feet from one end of the object and 12 feet from the other end?
7. Two forces of 20 lb. and 30 lb. have an included angle of 62°. Find the magnitude of their resultant force.

8. The distances from Portland, Oregon, to Mt. St. Helen and Mt. St. Jefferson are 53 miles and 74 miles, respectively. The angle between their directions is 115° 30′. Find their distance apart.

9. Two forces of 60 lb. and 80 lb. have a resultant of 100 lb. Find their included angle.

10. Two forces of 15 lb. and 25 lb., respectively, combine so as to form a resultant force of 30 lb. Find the angle that the resultant makes with each of its components.

11. The two diagonals of a parallelogram are 20 and 24, and they form an angle of 72°; find the sides.

12. To find the length of a horizontal tunnel AB through a hill, lines $AC = 860$ feet and $BC = 990$ feet were measured, also $C = 71°$. Find AB.

13. At a given time the distances of the earth and Venus from the sun were 92 000 000 and 66 000 000 miles, respectively. The angle ESV between their directions was 68° 40′. Find EV, the distance from the earth to Venus.

14. A road is to be constructed between two points A and B on opposite sides of a dense forest. To determine the length of the road a line is run from A to a point P and from P to B and the following measures taken: $AP = 870$ yds., $PB = 820$ yds., angle $APB = 72° 30′$. Find the length of the road.

15. A pole 13 feet long is placed 6 feet from the base of an embankment, and reaches 8 feet up its face. Find the slope of the embankment.

16. To find the distance between two church spires A and B a third point C is chosen. The distance AC is found to be 1000 feet and CB 1100 feet. The angle ACB is 62° 41′. Find the distance AB.

CHAPTER V

STRAIGHT LINE FORMULAS

82. Introduction. In Chapter I we have seen that the relation between two variables may be represented by a graph. This notion gives rise to two problems of fundamental importance.

1. *Given the equation of a graph, to draw the graph.*
2. *Given the graph, to find its equation.*

In some instances, the relation between two variables is represented by a straight line. For example, if coal costs \$8.00 per ton, then the total cost is \$8 times the number of tons, that is, $c = 8n$. If a man travels at the rate of 5 miles per hour, the distance he travels is 5 times the number of hours, that is, $d = 5n$. In general, if x and y are used to represent these variables, the relation between them, expressed in algebraic language, is $y = mx$, or $y = mx + b$.

83. Linear Equations and their Graphs. The equation

$$y = mx \tag{1}$$

is equivalent to the proportion

$$\frac{y}{m} = \frac{x}{1}. \tag{2}$$

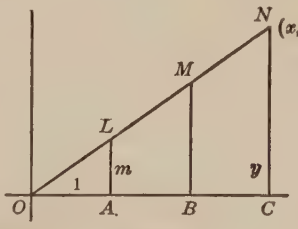


FIG. 57.

In Fig. 57, AL represents the slope (that is, cost per ton, distance per hour, etc.), for when $x = 1, y = m$. BM may be taken to represent the cost of 2 tons, distance traveled in 2 hours, etc. The triangles OAL, OBM, and OCN are similar. Hence

$$\frac{CN}{AL} = \frac{OC}{OA}, \quad \text{or} \quad \frac{y}{m} = \frac{x}{1}.$$

This similarity of triangles holds only when L is on the straight line ON, and the proportion in (2) holds only when the equation is of the first degree. Hence every equation of the form

$$y = mx$$

represents a straight line.

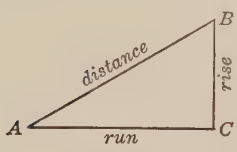


Fig. 58.

84. Rise, Run and Distance. Calculations of distance in a plane are usually made by means of right triangles whose vertical and horizontal sides are known. Thus, in speaking of a rafter, the horizontal and vertical distances (Fig. 58) are called the run and rise from one point to another.

Run from A to B = abscissa of B minus abscissa of A.

Rise from A to B = ordinate of B minus ordinate of A.

When the run from A to B is negative, B is to the left of A.

When the rise from A to B is negative, B is below A.

Distance between A and $B = \sqrt{(\text{run})^2 + (\text{rise})^2}$. Note that run and rise are sign-bearing, but that distance is always taken positive.

85. Distance between Two Points. Let it be required to find the distance between the points (1, 2) and (5, 5).

Solution. The rise is $5 - 2$ or 3; the run $5 - 1$ or 4. Hence the distance is

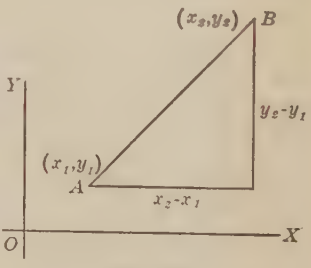


Fig. 59.

$$\sqrt{(5 - 1)^2 + (5 - 2)^2} = \sqrt{4^2 + 3^2} = 5.$$

In Fig. 59, the rise is $y_2 - y_1$ and the run is $x_2 - x_1$. Hence by the Pythagorean theorem, the distance AB between

$A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{3}$$

The distance between the points (4, 3) and (5, − 2) is

$$d = \sqrt{(4 - 5)^2 + (3 + 2)^2} = \sqrt{26}.$$

86. Mid-point of a Line. If the middle point of a straight line connecting two given points is to be found, its coordinates will be half the sum of the corresponding coordinates of the given points.

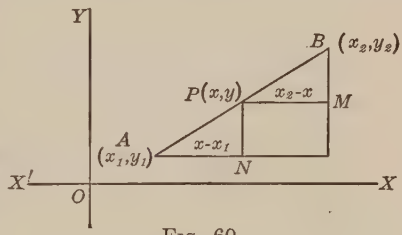


Fig. 60.

Proof. If P is taken as the mid-point of the line AB, Fig. 60, triangle ANP is congruent to triangle PMB, and $AN = PM$, also $NP = MB$. Hence, we may write

$$x - x_1 = x_2 - x.$$

Solving for x, we have

$$x = \frac{x_1 + x_2}{2}.$$

Similarly,

$$y = \frac{y_1 + y_2}{2}.$$

EXERCISES

1. Find the length of the line joining: (*a*) (2, 1) and (6, 7); (*b*) (3, 5) and (8, 11); (*c*) (0, 1) and (4, 7); (*d*) (2, 0) and (8, 5).

2. Find the distance between the points: (*a*) (1, 3) and (7, 5); (*b*) (2, −1) and (5, 3); (*c*) (2, 3) and (7, 10).

3. Find the length of the line joining: (*a*) (2, 3) and (8, 11); (*b*) (0, 4) and (3, 8); (*c*) (1, 4) and (6, 16); (*d*) (2, 5) and (4, 9); (*e*) (a, b) and (c, d); (*f*) (x_1, y_1) and (x_2, y_2).

4. The water distributing center for a town is at (0, 0). It gets the water from a reservoir at (12, 16). Find the length of the water main, allowing 5% extra for grades.

5. In the triangle whose vertices are (2, 3), (7, 5), and (5, 8), which is the longest side?

6. Show by the distance formula that the diagonals of the rectangle whose vertices are (2, 3), (11, 3), (2, 7), and (11, 7) are equal.

7. Find the length of each side of the quadrilateral whose vertices are (1, 2), (6, 6), (8, 11), and (3, 7). What geometric figure is it?

8. A quadrilateral has as vertices (2, 3), (10, 9), (7, 13), and (−1, 7). What geometric figure is it?

9. Find the coordinates of the mid-point of the line joining: (*a*) (2, 4) and (6, 8); (*b*) (0, 0) and (6, 8); (*c*) (3, 5) and (7, 11); (*d*) (a, b) and (c, d).

10. Find the length of the medians of a triangle whose vertices are (1, 4), (7, 6), and (5, 12). Define a median.

11. Find the mid-point of the hypotenuse of the right triangle whose vertices are (0, 0), (24, 0), and (0, 18). Show that this point is equally distant from all three vertices.

12. In the triangle whose vertices are (2, 3), (8, 7), and (6, 11) show that the line joining the mid-points of any two sides of the triangle is equal to one-half the remaining side.

13. Show that the points (1, 4), (3, −1), and (5, 4) are the vertices of an isosceles triangle.

14. Show that the points (2, 2), (−2, −2), and $(2\sqrt{3}, -2\sqrt{3})$ are the vertices of an equilateral triangle.

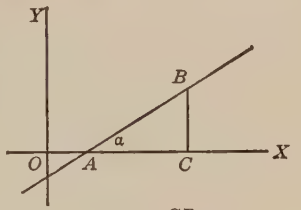
15. Show that (3, 0), (6, 4), and (−1, 3) are the vertices of a right triangle.

16. Show that (11, 2), (6, −10), (−6, −5) and (−1, 7) are the vertices of a square.

17. Find the lengths of the sides and medians of the triangle whose vertices are (2, 6), (7, −6), and (−5, −1). What kind of triangle is it?

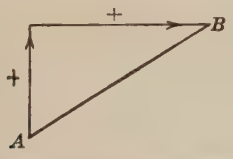
87. Inclination. As was stated on page 33, the angle which a line makes with the positive direction of the x-axis when the line is given the upward direction is called the *inclination* of the line or simply the inclination, and is frequently designated by α (alpha).

88. The Grade or Slope. The *grade* or *slope* of a straight line AB (Fig. 61) has been defined in § 39. It will be seen that it is also the rise from A to B divided by the run from A to B. If a straight line rises 6 feet in a horizontal distance of 100 feet, the grade, or slope, is said to be 6 in 100, or 0.06, or 6%. Note that the slope between A and B will be the same (both numerator and denominator changing sign) if we take both rise and run from B to A instead of from A to B. It will not do, however, to take A first in getting the rise, and B first in getting the run. Slope is a sign-bearing quantity; note the two cases.

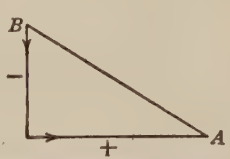


$$\text{Grade or Slope} = \frac{CB}{AC} = \tan\alpha = m$$

Fig. 61.



Slope is plus if AB (or BA) rises to the right.



Slope is minus if AB (or BA) falls to the right.

Fig. 62.

89. Slope of a Straight Line. From the definition of slope it follows that:

$$(4)\quad m = \tan\alpha = \text{slope of line} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{the difference of the ordinates}}{\text{the difference of the abscissas}}.$$

Exercise. Find the slope of the line joining the points (2, 3) and (7, 10).

Solution. Slope $= m = \dfrac{10 - 3}{7 - 2} = \dfrac{7}{5}$.

If the line rises to the right, the expressions $y_2 - y_1$ and $x_2 - x_1$ have the same sign, and the slope is positive. If the line falls to the right, the expressions $y_2 - y_1$ and $x_2 - x_1$ have opposite signs, and the slope is negative.

EXERCISES

Draw figures carefully in all cases.

1. Find the slope of the line through the points (1, 2) and (5, 8); through the points (2, 3) and (7, 10).

2. Find the slope of the line through the points (3, 1) and (6, 9); through the origin and (8, 5).

3. Find the slope of a line through the points (3, 0) and (0, 5); through the points (2, 1) and (−5, 6).

4. Find the slope of a line through the points (2, 0) and (5, 4); find also the values of $\sin \alpha$ and $\cos \alpha$ for this line.

5. What is the slope of the line $2x - y = 4$?

SUGGESTION. In drawing the graph, you get two points on the line; hence its slope can be found.

6. What is the slope of the line $y = 3x + 4$?

7. What is the slope of the line $y = \frac{1}{2}x + 2$?

8. Can you discover an easier way of finding the slope of a line when the equation is given than by the suggestion in exercise 5?

9. What is the slope of the line $y = 3x + 6$?

10. What is the inclination of the line joining (1, 3) and (5, 7)?

11. What is the inclination of the line joining (0, 4) and $(\sqrt{3}, 5)$?

12. What is the inclination of the line joining (5, 0) and $(6, \sqrt{3})$?

13. Find the slope of the line joining $(a + b, c + a)$ and $(c + a, b + c)$.

14. Find the slopes of the sides of a triangle whose vertices are (1, 2), (3, 6) and (7, 4).

15. Prove by means of slopes that (2, 3), (1, −3), and (3, 9) lie on the same straight line.

16. Prove that (5, 9) is on the line joining (2, 3) and (8, 15) and is equidistant from them.

17. Prove that the points $(a, b + c)$, $(b, a + c)$, and $(c, a + b)$ lie on the same straight line.

18. What is the inclination of a line parallel to $X'X$? Perpendicular to $X'X$?

19. What is the slope of a line parallel to $X'X$?

20. Given that the slope $= \tan \alpha = 5/12$. Find $\sin \alpha$ and $\cos \alpha$.

90. The Equation of a Line. A straight line is fixed definitely in position if it passes through a fixed point in a given direction, or if it passes through two fixed points. In the first case the slope is given, and in the second it must be found from the slope formula (4). There are several forms of the straight line equation, but for a particular line each one may in general be reduced to any of the others.

91. Point-Slope Form: $y - y_1 = m(x - x_1)$. Let (x_1, y_1) be a fixed point on the line whose slope is m (Fig. 63),

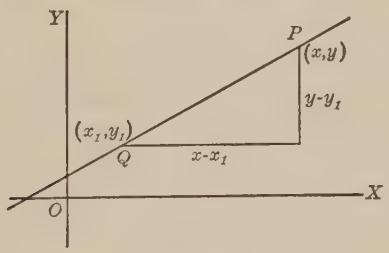


FIG. 63.

and (x, y) any other point. Then

$$\frac{y - y_1}{x - x_1} = m. \tag{5}$$

Clearing of fractions, we have

$$y - y_1 = m(x - x_1), \tag{6}$$

which is the equation of a line whose slope is m and which passes through the point $Q(x_1, y_1)$. This formula may be taken as fundamental.

92. Slope-Intercept Form: $y = mx + b$. If in the point-slope form the point (x_1, y_1) is identical with $(0, b)$, then b is the length of the intercept on the y-axis, and

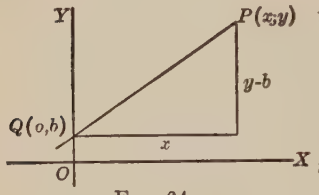


Fig. 64.

$$\frac{y - b}{x} = m, \tag{7}$$

(Fig. 64). Clearing of fractions and transposing b, we find

$$y = mx + b. \tag{8}$$

93. Line Through the Origin: $y = mx$. In equation (6), if both x_1 and y_1 are zero, the resulting equation is $y = mx$, the equation of a straight line through the origin with the slope m. (See § 83.)

EXERCISES

1. Determine the type form of each of the following equations. Name the two conditions given. Use that knowledge in drawing the graph.

(*a*) $y - 3 = 2(x - 5)$,	(*d*) $y - 1 = 4(x - 6)$.	(*g*) $y = \frac{3}{2}x$.
(*b*) $y = 2x + 7$.	(*e*) $y = 2x$.	(*h*) $y = x + 5$.
(*c*) $y = 3x - 5$.	(*f*) $y - 2 = 3(x - 4)$.	(*i*) $y - 1 = \frac{2}{3}(x - 5)$

2. State the equations which represent the following conditions. Do not simplify.

(*a*) Straight line through (2, 3) with slope 5.
(*b*) Straight line through (4, 1) with slope 2.
(*c*) Straight line through (0, 0) with slope 3.
(*d*) Straight line through (0, 5) with slope $\frac{3}{4}$.
(*e*) Straight line through (6, 2) with slope -2.
(*f*) Straight line through (0, 3) with slope -1.

3. State the equations which represent the following conditions. Put the equation in the form $Ax + By = C$.

(*a*) Straight line through (3, 5) with slope 2.
(*b*) Straight line through (4, -2) with slope 3.
(*c*) Straight line through (-2, -4) with slope 4.

(*d*) Straight line through $(-5, 4)$ with slope -3.
(*e*) Straight line through $(0, b)$ with slope m.
(*f*) Straight line through $(0, 0)$ with slope m.

4. Find the equations of the lines for which

(*a*) $m = 2, \quad b = 5.$ (*c*) $m = \frac{1}{2}, \quad b = 7.$
(*b*) $m = \frac{3}{4}, \quad b = -2.$ (*d*) $m = 2, \quad b = -8.$

5. Find the equation of a line through $(1, 5)$ and making an angle of $60°$ with the x-axis.

6. Find the equation of a line through $(2, -4)$ and making an angle of $30°$ with the x-axis.

7. Find the equation of a line through $(-3, 6)$ and making an angle of $45°$ with the x-axis. $30°$ with the y-axis.

8. What is the slope of a line bisecting the first and third quadrants? Second and fourth quadrants?

9. Find the equation of a line parallel to the y-axis and 3 units to the right of it. Two units to the left of it.

10. Find the equation of a line parallel to the x-axis and 4 units above it. Five units below it.

11. Find the equation of a line which intercepts the y-axis 4 units above the origin and makes an angle of $30°$ with the x-axis.

12. A medicine is 60% alcohol. Construct a graph showing the amount of alcohol in any given amount of the medicine, and find the equation of the graph.

13. The population of a town in 1918 was 1600, and in 1928, 2400. Assuming that growth in population can be represented by a straight line, construct the graph showing the population at any time from 1918 to 1928. Extend the graph and estimate from it the population for 1930, 1936, 1938.

14. The atmospheric pressure at sea-level is 15 lb. per square inch, and, for small changes of altitude, its decrease is proportional to the altitude (very nearly). If the decrease in the pressure is 0.06 lb. for each 100 ft. of ascent, find the equation connecting pressure and altitude and determine the difference in the readings at two places having a difference in altitude of 1000 ft.

15. A common application of the formula $y = mx + b$ is in finding the amount of a sum of money put at simple interest for a given time. The formula is $A = ni + P$, where A is the amount, i the rate of interest, expressed as a decimal, n the number of years, and P the prin-

cipal. Find the amount of \$75 at 8% simple interest for 12 years. Draw the graph.

16. A section of a stairway is shown in Fig. 65. It starts 2′ 6″ above the floor and has a rise of 6″ in a run of 8″. Will it clear a beam 10′ 8″ above the floor, and at a horizontal distance of 12′ from the foot of the stairway?

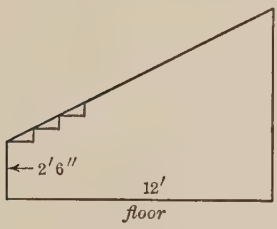


Fig. 65.

17. If C and F represent the readings on Centigrade and Fahrenheit thermometers, respectively, the relation between the two is given by

$$F° = \frac{9}{5}C° + 32°.$$

Construct the graph of the equation and show how to use it in changing from Centigrade to Fahrenheit readings and vice versa. What is the slope of this line? What is the F intercept? The C intercept?

18. The velocity V of sound, measured in feet per second, is given by the formula $V = 1090 + 1.14(t - 32°)$, where t is the temperature of the air in Fahrenheit degrees. Find (a) the velocity when the temperature is 32°, 50°, 60°, 80°, 95°, 100°, (b) the temperature when the sound travels 1120 feet per second.

19. When the water above a dam is at low water mark it exerts a pressure of 3 tons on the turbine. This pressure increases ½ ton for each additional foot of rise of the water. Draw a graph from which the pressure for any depth may be read.

20. Hooke's law for the extension of a stretched spring is $E = kT$, where E is the extension due to the tension or pull T. Represent the relation graphically, when $k = 2$. When $k = \frac{1}{2}$.

21. A head of 100 feet of water causes a pressure at the bottom of 43.43 pounds per square inch. Draw a graph showing the relation between head and pressure for all heads of water from 0 to 150 feet.

94. Two-Point Form: $(y - y_1)/(y_2 - y_1) = (x - x_1)/(x_2 - x_1)$. Let the two given points be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, and $P(x, y)$ be any other point on the line joining P_1P_2. Draw PA and P_2C perpendicular to OX, and PB and P_1C parallel to OX. Since the triangles PAP_1 and P_2CP_1 are similar,

we have $AP/CP_2 = P_1A/P_1C$, or

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}, \tag{9}$$

which may be written in the form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{10}$$

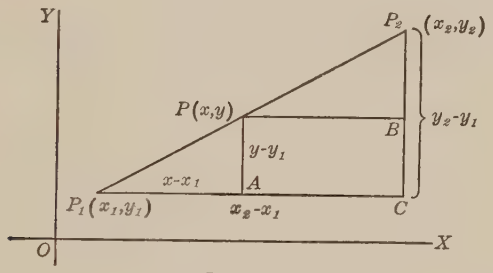


Fig. 66.

Since $(y_2 - y_1)/(x_2 - x_1) = m$, this form is equivalent to (6), § 91.

95. Two-Intercept Form: $x/a + y/b = 1$. If the points at which a line crosses the x-axis and the y-axis are denoted by $(a, 0)$ and $(0, b)$ respectively, and (x, y) is any point on the line, then from (9)

$$\frac{y - b}{0 - b} = \frac{x - 0}{a - 0}, \tag{11}$$

or

$$\frac{x}{a} + \frac{y}{b} = 1, \tag{12}$$

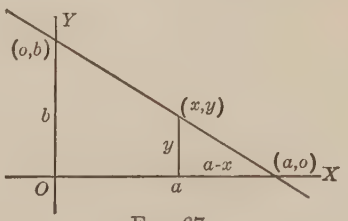


Fig. 67.

where a and b are the intercepts on the x and y axes, respectively.

96. The General Form of the Equation of the First Degree. Any linear relation between two variables x and y can be written in the form $Ax + By + C = 0$, where A, B and C are arbitrary constants, and A and B are not both zero.

It is evident that A and C or B and C; or A, B, or C may be zero, but not both A and B, for then C would be zero, and only the identity $0 = 0$ would be left.

If $B = 0$, the equation takes the form $Ax + C = 0$, or $x = -C/A$, the equation of a line parallel to the y-axis.

If $B \neq 0$, by solving for y, we obtain

$$y = -\frac{A}{B}x - \frac{C}{B}, \tag{13}$$

which is the equation of a line with slope $m = -A/B$, and y intercept, $b = -C/B$. Therefore in every case, when A and B are not both zero, the equation

$$Ax + By + C = 0 \tag{14}$$

represents a straight line.

EXERCISES

1. Determine the type form of each of the following equations. Name the two conditions given. Use that knowledge in drawing the graph.

(*a*) $y = 2x + 3$.

(*b*) $y - 3 = 2(x - 4)$.

(*c*) $y = 3x$.

(*d*) $\frac{x}{4} + \frac{y}{3} = 1$.

(*e*) $\frac{y-2}{7-2} = \frac{x-3}{8-3}$.

(*f*) $\frac{y-3}{5-3} = \frac{x-1}{6-1}$.

(*g*) $\frac{x}{6} + \frac{y}{2} = 1$.

(*h*) $\frac{y-4}{10-4} = \frac{x-3}{7-3}$.

(*i*) $\frac{x}{6} + \frac{y}{3} = 1$.

2. Write the equations of the lines passing through the points.

(*a*) (2, 4), (7, 10).
(*b*) (3, 1), (6, 9).
(*c*) (−2, −3), (4, 5).
(*d*) (−3, 4), (2, 7).
(*e*) (−2, −5), (4, 6).
(*f*) (0, 0), (8, 11).

3. Find the equations of the lines having the intercepts.

(*a*) $a = 7, b = 5$.	(*d*) $a = 6, b = -4$.	(*g*) $a = -4, b = -5$.
(*b*) $a = 6, b = 4$.	(*e*) $a = -3, b = 7$.	(*h*) $a = -5, b = -2$.
(*c*) $a = 5, b = -3$.	(*f*) $a = -8, b = 4$.	(*i*) $a = a, b = b$.

4. Write the equation of:

(*a*) Straight line through (2, 3) and (6, 8).
(*b*) Straight line through (1, 4) and (7, 10).
(*c*) Straight line through (3, 1) and (0, 9).
(*d*) Straight line through the origin and (4, 5).
(*e*) Horizontal line through (4, 7).
(*f*) Vertical line through (5, 9).
(*g*) Straight line through (3, 5) and (6, 10).
(*h*) Straight line through (a, 0) and (0, b).

5. A straight line passes through the points (3, 5) and (8, 12). Find its intercept on (*a*) the x-axis, (*b*) the y-axis.

6. A certain spiral spring is stretched to the lengths 3.6 inches and 4.4 inches by weights of 8 lb. and 12 lb., respectively. Assuming that the varying weights w and the corresponding lengths l are connected by a linear relation, find that relation and draw the graph.

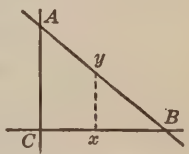


FIG. 68.

7. A straight section of a railroad intersects two roads which meet at right angles cutting off the triangle ACB. $AC = 46$ rods, $CB = 64$ rods. Find the ordinate xy corresponding to the following abscissa: $Cx = 18$ rods, $Cx = 28$ rods, $Cx = 54$ rods. Compute the acreage in the trapezoid in each case.

8. In a survey a straight line passes through the points (coordinates in rods), (2, 3) and (14, 30). Will it pass through the points whose coordinates are (40, $88\frac{1}{2}$)? (4, $7\frac{1}{2}$)? (3, $5\frac{1}{4}$) miles?

9. A straight line is drawn so as to cross the x-axis at (6, 0) and the y-axis at (0, 5). Will it pass through the point (8, $-1\frac{2}{3}$)? (3, $2\frac{1}{2}$)? (−12, 15)? (−30, 30)?

10. Discuss the possible uses of the straight line formulas (*a*) in a tile or drainage survey; (*b*) in building plans; (*c*) in plans for grading roads and streets; (*d*) in designs.

97. Parallel Lines. Draw two non-vertical parallel lines as shown in Fig. 69. Select any two points on each line. The slopes are, respectively, a_1/b_1 and a_2/b_2. Since the triangles are similar the slopes are equal, and $a_1/b_1 = a_2/b_2$. That is, if two non-vertical lines are parallel, their slopes are equal.

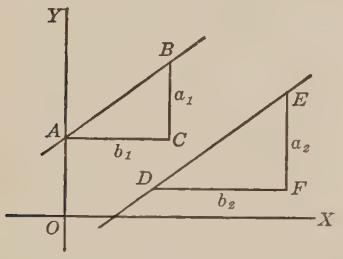


Fig. 69.

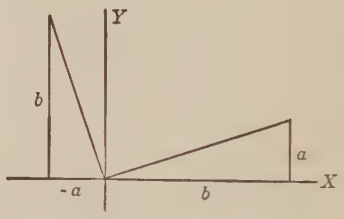


Fig. 70.

98. Perpendicular Lines. Draw any two perpendicular lines through the origin, not parallel to the axes (Fig. 70). On these lines lay off equal distances from the origin, and drop perpendiculars to the x-axis. Let the slope of one line be $m_1 = a/b$. Then the slope of the other is $m_2 = -b/a$. (Since the triangles are equal the corresponding sides are equal.) Hence

$$m_1 m_2 = \frac{a}{b}\left(-\frac{b}{a}\right) = -1, \tag{15}$$

or

$$m_1 = -\frac{1}{m_2}. \tag{16}$$

Hence, if two lines are perpendicular to each other, and not parallel to the axes, the slope of one is the negative reciprocal of the slope of the other. Another statement of the same fact is that the product of the two slopes is -1.

EXERCISES

1. Draw the graphs of $y = 2x - 2$, $y = 2x$, and $y = 2x + 4$. What do you notice about the graphs? Compare the slopes.

2. Draw the graphs of $y = 2x - 4$ and $y = -\frac{1}{2}x + 2$. What do you notice about the graphs? Compare the slopes.

3. What change must be made in the equation $y = 3x - 1$ if every point in its graph is to be raised vertically 2 units? 5 units? Lowered 7 units?

4. Given the line $y = 2x + 5$, what angle does it make with the x-axis?

SUGGESTION. The tangent of the angle equals the slope.

5. Find the angle which the straight line joining the following points makes with the x-axis.

(*a*) (1, 3) and (5, 7).
(*b*) (2, 5) and (3, 8).
(*c*) (0, 0) and (6, 6).
(*d*) (−3, −2) and (4, 9).
(*e*) (−3, −5) and (−1, −2).
(*f*) (6, 0) and (0, 4).

6. Find the equations of lines satisfying the following conditions:

(*a*) through (2, 5) and parallel to $y = 3x + 2$.
(*b*) through (3, 4) and parallel to $2x - 3y = 10$.
(*c*) through (1, 6) and perpendicular to $y = -\frac{3}{4}x + 7$.
(*d*) through (2, 3) and perpendicular to $3x - 4y = 8$.
(*e*) through (1, 5) and perpendicular to $y = 2x - 10$.
(*f*) through (5, 4) and perpendicular to $5x - 2y = 9$.

7. In a drawing, the line representing the slope of a hillside has the equation $y = \frac{1}{2}x + 1$. Find its inclination.

8. Construct a line through the point (4, 3) whose slope is 2. Find its equation.

9. The approach to a bridge has a rise of 4 feet in a horizontal distance of 20 feet. Find the equation of the approach if it starts 10 ft. above the datum plane.

10. By means of slopes, show that the diagonals of a square are perpendicular to each other.

11. A line passes through (1, 3) and (5, 8). What is its slope? What is the slope of a line perpendicular to it?

12. What is the slope of a line whose equation is $y = \frac{2}{3}x + 7$? What is the slope of a line perpendicular to it?

13. Find the slope of each of the following lines:

(*a*) $y = 2x + 7$. (*c*) $3x - 5y = 8$. (*e*) $\frac{x}{2} + \frac{y}{3} = 1$.

(*b*) $y = -\frac{1}{3}x + 4$. (*d*) $2x + 7y = 10$. (*f*) $3y - \frac{x}{2} = 5$.

14. Find the equation of a line through the point $(1, -5)$

(*a*) parallel to $y = 5x + 4$;

(*b*) perpendicular to $y = \frac{1}{7}x + 3$.

15. Is the line joining (1, 3) and (6, 10) perpendicular to the line joining (2, 0) and (−5, 5)?

16. The vertices of a quadrilateral are (4, 2), (16, 10), (6, −1) and (18, 7). Is it a rectangle?

17. The vertices of a quadrilateral are (2, 3), (9, 2), (12, 7) and (8, 10). Show that the lines joining the mid-points of its sides in order form a parallelogram.

18. Show that (−1, −2), (3, 2) and (−3, 0) are the vertices of a right triangle.

19. Where does a line cut the x-axis if it passes through (2, −3) and is parallel to the line through (−1, 5) and (4, −2)?

20. Show that the equation of the tangent to the circle

$$x^2 + y^2 = r^2$$

at the point (x_1, y_1) is

$$xx_1 + yy_1 = r^2.$$

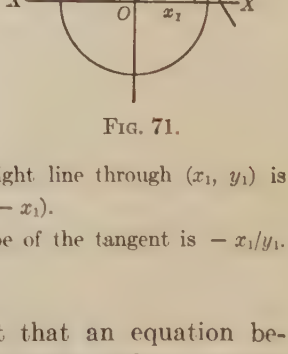


FIG. 71.

SUGGESTION. The equation of a straight line through (x_1, y_1) is

$$y - y_1 = \text{slope } (x - x_1).$$

The slope of OP_1 is y_1/x_1; hence the slope of the tangent is $-x_1/y_1$. In reducing the equation remember that

$$x_1^2 + y_1^2 = r^2.$$

99. Locus of a Point. The fact that an equation belongs exclusively to a particular curve makes possible the solution of many geometric problems by means of coordinates. This method was devised by Descartes and published by him in 1637. In honor of him the system of

coordinates used in Chapters I and V is called Cartesian coordinates.

The word *locus* means place or position. The locus of a moving point subject to a given condition is the totality of positions where a point may lie and satisfy the given conditions.

For example, in plane geometry it is proved that the perpendicular bisector of the line segment joining two fixed points is the locus of all points equidistant from these points.

DEFINITION. An equation which is true for the coordinates of any point on a curve, or locus, but is not true for the coordinates of any other point, is called the equation of the curve, or the equation of the locus. Conversely, the curve is called the curve or locus of the equation.

100. Equation of the Locus. The following steps will be found useful in finding the equation of the locus of a point satisfying a given condition.

FIRST. Assume that $P(x, y)$ is any point satisfying the given condition and is therefore on the locus.

SECOND. Write down the given condition.

THIRD. Express the given condition in terms of x and y and simplify the result. This will be the required equation.

EXAMPLE 1. The point $P(x, y)$ moves so that it is always equidistant from $A(1, 2)$ and $B(5, 8)$. Find the equation of the curve.

SOLUTION. FIRST STEP. Let $P(x, y)$ be any point on the curve. Then by the given condition,

SECOND STEP. $PA = PB$.

THIRD STEP. Making use of the formula for the distance between two points, we obtain

$$PA = \sqrt{(x-1)^2 + (y-2)^2},$$

$$PB = \sqrt{(x-5)^2 + (y-8)^2}.$$

Hence, by step two,

$$\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x-5)^2 + (y-8)^2}.$$

Squaring and reducing, we obtain

$$2x + 3y = 21.$$

Example 2. Find the equation of a point which moves so that its distance from (16, 0) is always twice its distance from (4, 0).

Solution. First Step. Let $P(x, y)$ by any point on the locus. Then by the given condition, we find (Fig. 72)

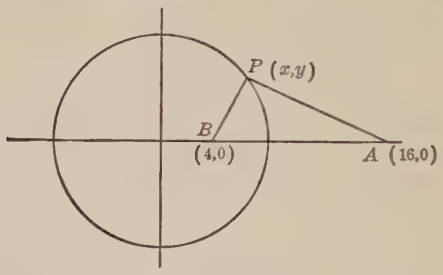


Fig. 72.

Second Step. $PA = 2PB$.

Third Step. $PA = \sqrt{(x-16)^2 + y^2}$, $\quad PB = \sqrt{(x-4)^2 + y^2}$.

Hence, by step two,

$$\sqrt{(x-16)^2 + y^2} = 2\sqrt{(x-4)^2 + y^2}.$$

Squaring and collecting terms and dividing by 3, we obtain as the required equation: $x^2 + y^2 = 64$.

EXERCISES

1. Find the equation of the curve, each of whose points is equidistant from

(*a*) the points (2, 3) and (6, 9).
(*b*) the points (1, 4) and (−5, 10).
(*c*) the points (−4, −6) and (2, 10).
(*d*) the points (−4, 2) and (6, −8).

2. Find the equation of the perpendicular bisector of the line-segment joining (2, −3) and (6, 7).

3. Find the equation of the perpendicular bisector of the line-segment joining (0, 4) and (−4, 6).

4. Find the equation of the curve, all points of which are

(*a*) 6 units from (3, 4).
(*b*) 8 units from (−4, −1).
(*c*) 5 units from (5, 5).
(*d*) 7 units from (1, −6).
(*e*) 24 units from (3, −5).
(*f*) 17 units from (−6, 7).

5. Where are all the points for which $y=x$? $y=x^2$? $x^2+y^2=49$?

6. Which of the following points are on the curve $y = x^2 + 2x + 3$: (2, 11), (0, 3), (1, 6), (3, 0), (5, 37)?

7. What must be the value of r in the equation $x^2 + y^2 = r^2$ if the curve passes through (7, 2)?

8. Show analytically that the locus whose points are all equidistant from (1, 3) and (6, −4) is the perpendicular bisector of the line-segment joining these points.

9. A point moves so that its distance from the y-axis is always equal to its distance from (4, 0). Find the equation of its path.

10. A point moves so that its distance from the y-axis is always equal to its distance from (6, 0). Find the equation of its path.

11. A point moves so that its distance from the line $x = -4$ is always equal to its distance from (4, 0). Find the equation of its path.

12. A line is drawn perpendicular to the line through P_1 (−2, 5) and P_2 (4, −3) at its middle point. Find a point P on this perpendicular whose abscissa is 2. Show that P is equidistant from P_1 and P_2.

101. Historical Note. The branch of mathematics given in this chapter is known as analytic geometry. Its distinguishing characteristics are the employment of coordinates and the correlation of algebra and geometry. With the Hindus and also the Greeks, algebra and geometry were quite largely independent mathematical branches. A limited use of coordinates had continued since the time of Hipparchus (140 B.C.). However, the system of connecting points with pairs of numbers which refer to axes geometrically independent of the curves, and the use of variables to represent distances associated with the curves, was devised by Descartes (§ 99). The treatment of geometry by means of coordinates and equations proved to be so systematic and powerful that it permitted a tremendous extension of geometry. Soon the problem of finding the direction of a curve at any point arose, and the investigations which followed led to the invention of calculus by Sir Isaac Newton, about 1666.

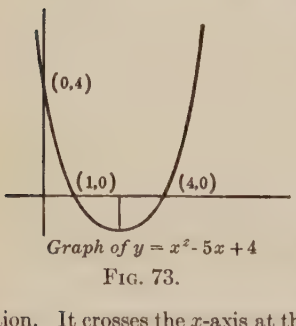
CHAPTER VI

THE QUADRATIC FUNCTION

102. Definition. A function which has the form $ax^2 + bx + c$, where a, b, and c are any given numbers, is called a *quadratic function.*

For example, $3x^2 - 5x$ and $\frac{2}{3}x^2 + 7x + \frac{4}{5}$ are quadratic functions of x.

The graph may be obtained from the equation by first tabulating a series of points. These points may then be plotted and a smooth curve drawn through them.



Graph of $y = x^2 - 5x + 4$

FIG. 73.

EXAMPLE. Construct the graph of $y = x^2 - 5x + 4$.

SOLUTION. Give to x the values 0, 1, 2, 3, 4, etc., and compute the corresponding values of y (the function). The curve through these points, taken in order, is the graph of the function. It crosses the x-axis at the points whose abscissas are 1 and 4 (see Fig. 73).

The values of x which satisfy an equation are called its ***roots.*** Hence the roots of the equation

$$x^2 - 5x + 4 = 0$$

are 1 and 4.

103. Solution by Factoring. Some quadratic equations can be solved by factoring, as follows.

1. *Transpose all terms to the left-hand member.*
2. *Factor the left-hand member.*
3. *Place each factor equal to zero, and solve for the variable.*

EXAMPLE 1. Solve $2x^2 - 7x + 5 = 0$.

Factoring, we have

$$x - 1 = 0, \quad \text{or} \quad 2x - 5 = 0,$$

whence $x = 1$ or $5/2$. Any value of x which makes either factor equal to zero will satisfy the equation. Hence 1 and $5/2$ are solutions.

CHECK. Substituting 1 for x in the equation, we obtain $2 - 7 + 5 = 0$. Substituting $5/2$ for x in the equation, we obtain $25/2 - 35/2 + 5 = 0$.

EXAMPLE 2. Solve $x^2 - 7x = 0$.

This may be written in the form $x(x - 7) = 0$. Setting each factor equal to zero, and solving for x, we obtain $x = 0$, $x - 7 = 0$. Hence the roots are 0 and 7.

EXERCISES

1. Draw the graph of each of the following equations and find the points of crossing the x-axis.

(a) $y = x^2 - 6x + 5$.
(b) $y = x^2 - 4$.
(c) $y = x^2 - 9$.
(d) $y = x^2 - 5x + 6$.
(e) $y = x^2 + 5x + 6$.
(f) $y = x^2 + 4x + 3$.
(g) $y = -(x^2 - 4x + 3)$.
(h) $y = x^2 + 2x$.

NOTE. The student should notice that the abscissas of the points where the graph crosses the x-axis are the roots of the equations formed by equating y to zero.

2. Determine without plotting which of the following points are on the parabola $y = x^2 - 6x + 8$: (0, 8), (3, −1), (4, −2), (2, 0), (5, 3), (6, 7), (4, 0), (3, 5), (−1, 14).

3. For what value of a will the curve $y^2 = 4ax$ pass through (4, 4)?

Solve each of the following quadratic equations by factoring.

4. $x^2 - 5x + 6 = 0$.
5. $x^2 - 7x + 12 = 0$.
6. $x^2 - 9x + 14 = 0$.
7. $3x^2 - 5x + 2 = 0$.
8. $2x^2 - 9x + 7 = 0$.
9. $x^2 + 11x + 30 = 0$.
10. $x^2 + 7x + 6 = 0$.
11. $2x^2 + 7x + 5 = 0$.
12. $2s^2 - s - 36 = 0$.
13. $9x^2 - 27x - 70 = 0$.
14. $(x - 3)^2 - 25 = 0$.
15. $2x^2 + x - 3 = 0$.
16. $3x^2 - 2x - 8 = 0$.
17. $6x^2 + 35x - 6 = 0$.
18. $4x^2 - 28x + 49 = 0$.
19. $x^2 - 6x + 9 = 0$.
20. $x^2 - 2ax + a^2 = 0$.
21. $x^2 - (a + b)x + ab = 0$.
22. $2x^2 - 3mx + m^2 = 0$.
23. $7s^2 + 10s - 8 = 0$.
24. $\sqrt{x + 5} = x + 3$.
25. $\sqrt{x + 10} = 3x - 14$.

SUGGESTION. If the equation contains a radical, put the radical on one side of the equality sign and all other terms on the other side. Square both members to remove the radical sign. Test each result by substituting that value of x in the original equation.

26. $\sqrt{x+16} = x - 4.$
27. $x + \sqrt{x+16} = 14.$
28. $\sqrt{2x-2} - \sqrt{34-x} = -1.$
29. $\sqrt{4x+13} - x = -2.$
30. $\sqrt{x^2+3x} + 2 = x + 1.$
31. $\sqrt{2x+5} + \sqrt{x-10} = 5.$
32. $\sqrt{x} = 8 - \dfrac{15}{\sqrt{x}}.$
33. $\sqrt{\dfrac{3x+28}{x+10}} = \sqrt{x+10}.$
34. $\sqrt{x(6+x)} + 2x = 8.$
35. $(x + 1/x)^2 - 3x - 3/x = 10.$
36. $\sqrt{2x+1} - x = -1.$
37. $\sqrt{4x-11} + \sqrt{2x-9} = 4.$

104. Solution by Completing the Square. A quadratic equation may be solved by the aid of a process known as *completing the square*. For example, to solve

$$2x^2 - 8x - 10 = 0,$$

write the equation in the form

$$x^2 - 4x = 5.$$

Adding 4, the number which completes the square of the left-hand member, to both sides, we obtain

$$x^2 - 4x + 4 = 9,$$

or

$$(x - 2)^2 = 9.$$

Taking the square root, we have

$$(x - 2) = \pm 3.$$

Transposing, we find

$$x = 2 \pm 3.$$

Whence

$$x = 5 \text{ or } -1.$$

CHECK. Substituting these values for x in the equation, we obtain $50 - 40 - 10 = 0$, and $2 + 8 - 10 = 0$.

105. The Quadratic Formula. Every quadratic equation in a single variable can be reduced to the standard form

$$ax^2 + bx + c = 0, \tag{1}$$

where a, b, and c are constants and $a \neq 0$. We can solve (1) by the method of completing the square.

SOLUTION. Transposing c, we find

$$ax^2 + bx = -c.$$

Dividing by a,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing the square, we have

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

Simplifying,

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root, and subtracting $b/2a$ from both members,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

Hence the roots of the general quadratic $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{3}$$

The formula (2) may be used for solving any quadratic equation by substituting for a, b, and c of the formula their values from the given equation.

EXAMPLE. Solve the equation $3x^2 - 14x + 8 = 0$. Here $a = 3$, $b = -14$, $c = 8$. Substituting in the formula, we find

$$x = \frac{14 \pm \sqrt{196 - 96}}{6} = \frac{14 \pm 10}{6},$$

whence

$$x = 4 \quad \text{or} \quad 2/3.$$

EXERCISES

In the following equations state what must be added to each side of the equation in order to complete the square of the left hand member. Solve the equations.

1. $x^2 - 4x = 12$.
2. $x^2 - 6x = 16$.
3. $x^2 + 2x = 3$.
4. $x^2 + 8x = 8$.
5. $4x^2 - 8x = 12$.
6. $4x^2 - 12x = 27$.
7. $4x^2 - 20x = 75$.
8. $9x^2 - 18x = 40$.
9. $x^2 - 3x = 10$.
10. $x^2 - 7x = 13/4$.
11. $x^2 - 5x = 6$.
12. $x^2 - 5x = 7/2$.

Solve the following equations by the use of the formula, and verify by substitution in the equations.

13. $2x^2 - 5x + 3 = 0$.
14. $2x^2 - 7x + 5 = 0$.
15. $3x^2 - 5x + 2 = 0$.
16. $4x^2 - 9x + 5 = 0$.
17. $6x^2 - 11x + 5 = 0$.
18. $2x^2 - 11x + 12 = 0$.
19. $3x^2 - 23x + 30 = 0$.
20. $3x^2 + 5x + 2 = 0$.
21. $7x^2 + 37x = -10$.
22. $4x^2 + 17x = -15$.
23. $6x^2 + 5x = -1$.
24. $s^2 + 12 = 8s$.
25. $s^2 + 2s = 35$.
26. $16t^2 + 15 = 34t$.
27. $5x^2 - 11x + 6 = 0$.

106. Nature of the Roots. Discriminant. The two roots of the quadratic equation (1) are given by (3), § 105. If a, b, and c are real numbers, the numerical character of the roots depends upon the number $b^2 - 4ac$, called the *discriminant* of the quadratic equation. From (3) it appears that:

(*a*) If $b^2 - 4ac > 0$, the roots are real and distinct; hence the graph cuts the x-axis in two distinct points.

(*b*) If $b^2 - 4ac = 0$, the roots are equal; hence the graph is tangent to the x-axis.

(*c*) If $b^2 - 4ac < 0$, the roots are imaginary; hence the graph does not touch the x-axis.

These three cases are illustrated in Fig. 74, which shows the graphs of the three functions

$$(a)\ x^2 - 2x, \quad (b)\ x^2 - 2x + 1, \quad (c)\ x^2 - 2x + 2.$$

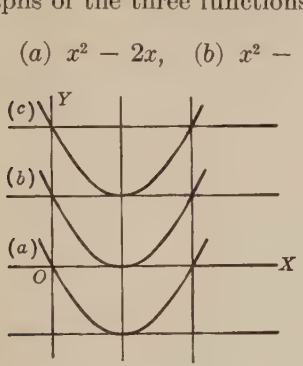


Fig. 74.

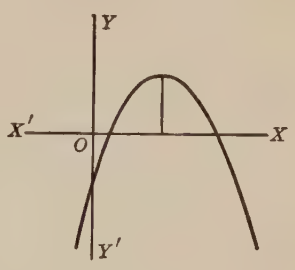


Fig. 75.

107. Coordinates of the Maximum or Minimum Point. If in the function $ax^2 + bx + c$ the coefficient of x^2 is positive, the graph resembles in form and direction the curve of Fig. 73. The number expressing the ordinate drawn to the lowest point of the graph is called the minimum value or least value of the function. The curve may or may not cross the x-axis. If the coefficient of x^2 is negative, the curve is concave downward and resembles in form and in direction the curve of Fig. 75. The number expressing the ordinate drawn to the highest point of the curve is called the maximum or greatest value of the function. Maximum and minimum values may be found approximately from the curve, or exactly by expressing the function as the sum or difference of two terms, the one containing x being a perfect square.

To illustrate, let the function $x^2 - 4x + 2$ be written in the form $(x - 2)^2 - 2$, in which the term $(x - 2)^2$ alone is variable. Since it is a square it is either a positive number or zero. Hence the least value of the function is the value that corresponds to the least value of $(x - 2)^2$, namely zero. When $(x - 2)^2 = 0$, $x = 2$ and the corresponding value

of the function is -2. Hence the coordinates of the minimum point on the curve are $(2, -2)$.

Similarly the function $-x^2 + 4x + 2$ may be written $-(x - 2)^2 + 6$. Since the first term is always negative or zero, the greatest value of the function corresponds to $-(x - 2)^2 = 0$. Hence the maximum value of the function is 6. The coordinates of the corresponding point are $(2, 6)$.

If the coefficient of x^2 is not unity, the function must first be put in the form

$$a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right).$$

The factor within the parentheses may then be transformed as before.

108. Interpretation of Roots. While every quadratic equation has two roots, the problem which leads to the equation may have two, one, or no solution, since the roots obtained may not admit of interpretation under the conditions stated.

(*a*) If the unknown quantity which represents a number of men comes out fractional, it indicates the impossibility of fulfilling the conditions, since the number of men must be integral.

(*b*) If the unknown represents an area or volume, a negative result indicates the impossibility of the given conditions being fulfilled by areas or volumes; if the unknown represents resources, a negative result indicates liabilities; if the unknown represents distance traveled, a negative result indicates distance in the opposite direction.

(*c*) Similarly, imaginary results sometimes admit of interpretation. In physics, especially electricity and magnetism, a thorough knowledge of imaginary and complex numbers is indispensable.

109. Historical Note. In the second and the sixth books of Euclid's Elements, the quadratic equation is solved

geometrically. The use of the geometric method rather than the algebraic method may be due to the fact that the Greeks did not consider arithmetic processes to be scientific work. Even Plato pronounced computation to be a vulgar and childish art beneath the dignity of a philosopher. Leonardo of Pisa in 1202 urged upon all students the importance of geometry as a means of proving algebraic rules and principles. Pacioli (1494) followed the method of Leonardo, as did also Regiomontanus (1436–76 A.D.), and his contemporaries. Since Diophantus and other Greeks did not have the fundamental algebraic conception of a negative number, they did not solve the general form of the quadratic. The Hindu mathematician Sridhara and other Hindus used a method of solving that is similar to the one we now use. Cardan (1501–76 A.D.) gave the correct interpretation of the negative number, making it possible to interpret the equation if the roots are any real numbers.

EXERCISES

1. Determine the character of the roots of each of the following quadratics.

(a) $3x^2 - 7x + 4 = 0$. (f) $3x^2 - 8x + 6 = 0$.
(b) $5x^2 - 18x + 16 = 0$. (g) $3x^2 - 12x + 12 = 0$.
(c) $3x^2 - 5x + 2 = 0$. (h) $x^2 - 2x + a^2 = 0$.
(d) $5x^2 + 12x + 7 = 0$. (i) $x^2 - 10x + 30 = 0$.
(e) $7x^2 + 13x + 6 = 0$. (j) $ax^2 + bx + c = 0$.

2. Determine the values of k so that the roots of the following equation may be equal.

$$3x^2 - kx + 12 = 0.$$

SOLUTION. In order that the roots of this equation may be equal, $b^2 - 4ac = k^2 - 144 = 0$, whence k must equal ± 12. Substituting these values in the equation, we get $3x^2 \pm 12x + 12 = 0$, or $(x \pm 2)^2 = 0$.

3. Determine the values of k so that the roots of each of the following quadratics may be equal.

(a) $x^2 - kx + 4 = 0$.
(b) $x^2 - 2kx + 36 = 0$.
(c) $4x^2 - 12x + k = 0$.
(d) $16x^2 - 56x + k = 0$.
(e) $kx^2 + 6x + 1 = 0$.
(f) $k^2x^2 + 10x + 1 = 0$.
(g) $k^2x^2 + 18x + 9 = 0$.
(h) $x^2 - 12x + 8k = 0$.
(i) $(4x + k)^2 = 16x$.
(j) $3x^2 + 6kx + 75 = 0$.

4. By completing the square, determine the nature of the roots and the minimum or maximum value of each of the following quadratics. Check these results from the graph.

(a) $y = x^2 - 4x + 4$.
(b) $y = x^2 - 5x + 4$.
(c) $y = x^2 - 9$.
(d) $y = x^2 - 20x + 50$.
(e) $y = x^2 + 4x$.
(f) $y = x^2 - 2x - 4$.
(g) $y = x^2 - 6x - 10$.
(h) $y = x^2 - 6x - 9$.
(i) $y = 5 - 4x - x^2$.
(j) $y = x^2 - 6x + 4$.

5. Find the maximum or minimum value, as the case may be, of each of the following functions. What are the coordinates of the maximum or minimum point of the graph?

(a) $x^2 - 4x + 5$.
(b) $x^2 - 2x + 8$.
(c) $x^2 - 6x + 3$.
(d) $-x^2 + 4x + 11$.
(e) $-x^2 - 2x + 7$.
(f) $-x^2 - 6x - 8$.
(g) $2x^2 + 8x + 10$.
(h) $x^2 + 5$.
(i) $x^2 - 7$.

6. What are the general characteristics of the graph of the function $ax^2 + bx + c$, if a is negative?

7. Find the dimensions and the area of the largest rectangle that can be enclosed by 16 rods of fence.

SUGGESTION. Let x represent one side of the rectangle in rods and y the area in square rods. Express y in terms of x and plot. Obtain the required values of x and y from the graph.

8. The height in feet of a body thrown vertically upward is given by the equation:

$$h = -16t^2 + 64t.$$

Find the maximum height to which it will rise.

9. A line 10 inches long is to be divided into two parts whose product is a maximum. Find the length of each part.

SUGGESTION. $y = 10x - x^2$.

10. Divide 10 into two parts such that the sum of their squares is a minimum.

11. Determine (without solving) the nature or character of the roots in each of the following quadratic equations.

(*a*) $x^2 - 5x + 4 = 0$. (*e*) $x^2 - 4x - 12 = 0$. (*i*) $x^2 - 6x + 1 = 0$.
(*b*) $x^2 + 5x + 6\frac{1}{4} = 0$. (*f*) $x^2 - 2x + 10 = 0$. (*j*) $3x^2 - 5x + 2 = 0$.
(*c*) $x^2 - 5x + 7 = 0$. (*g*) $x^2 - 5x - 14 = 0$. (*k*) $2x^2 + 7x + 5 = 0$.
(*d*) $x^2 - 6x + 9 = 0$. (*h*) $x^2 + 3x + 7 = 0$. (*l*) $7x^2 + 11x = 50$.

12. For what values of m will the roots of the quadratic equation $m^2x^2 + 8x + 1 = 0$ be equal?

13. In each of the following equations, can x be interpreted in terms of (*a*) men? (*b*) resources?

(*a*) $x^2 - 5x + 6 = 0$. (*c*) $x^2 - 5x - 24 = 0$. (*e*) $x^2 - 4x + 10 = 0$.
(*b*) $x^2 - 3x - 10 = 0$. (*d*) $x^2 + 4x + 4 = 0$. (*f*) $x^2 + 4x - 10 = 0$.

14. Find the area and dimensions of the largest rectangle that can be inscribed in a right triangle whose base is 8 inches and altitude 10 inches so that the right angle of the triangle shall be one of the angles of the rectangle.

SUGGESTION. Let A (area) $= xy$. By means of similar triangles express y in terms of x, then express A in terms of x.

110. The Parabola. The parabola is a particularly important curve. The path of any projectile, whether it be a baseball or a projectile from a cannon, is a parabola provided the path is not vertical, and provided the resistance of the air is neglected. This curve is frequently found in bridge construction, cross-sections of headlights, searchlights, reflecting telescopes, reflecting walls about the stage of an auditorium, the hollow surface of a rotating liquid, the "crown" of a pavement, and the graphs of many scientific formulas such as $D = gt^2/2$, $y = kx^2$, etc.

The Hell Gate bridge over East River, New York City, is one of the most beautiful bridges in the world. The arch, which is a true parabola, has a span of 977.5 feet and a height of 320 feet. The Williamsburg bridge over East River, New York City, is the longest suspension bridge in the world. The parabolic arch is 1600 feet long in span and has a depth of 118 feet. The weight of the span is 8000 tons.

The intimate connection between beauty of form and numerical relations has been studied since the beginning of precise geometric thinking. The beauty of the parabola and its useful properties have led to serious study of definitions and properties of such curves.

PRINCIPLE. Rays of light or heat (from the sun or a star) parallel to the axis of a parabolic surface converge to the focus of the parabola; conversely, rays (light, heat, or sound) which emanate from the focus are reflected parallel to the principal axis. This principle is very commonly used; it is found in auditoriums, reflecting mirrors, headlights, etc.

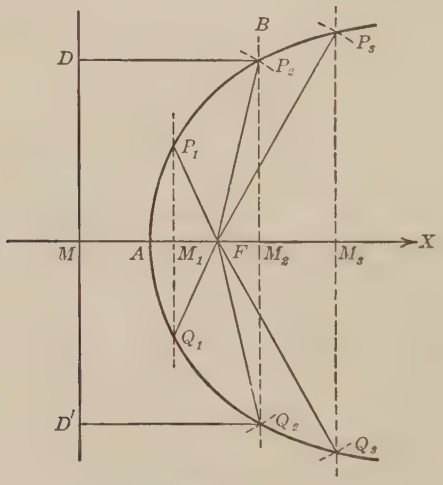


FIG. 76.

111. Definition of a Parabola. *A* ***parabola*** *is the locus of a point whose distance from a fixed point is always equal to its distance from a fixed line.* The fixed point is called the ***focus,*** and the fixed line the ***directrix,*** of the parabola.

112. Mechanical Construction. (*a*) *By points.*

Given the focus F and the directrix DD', Fig. 76. Drop a perpendicular from F to DD', meeting it at M. Let A be the middle point of FM. Then, by definition, A is a point on the curve. Draw lines perpendicular to FM at M_1, M_2, M_3, etc. With F as a center and MM_1 as a radius, strike an arc cutting the perpendicular through M_1 in two points P_1 and Q_1; similarly with F as a center and MM_2 as a radius, draw an arc cutting the perpendicular through M_2 at P_2 and Q_2, and so on. These points satisfy the definition of a parabola. If they are connected by a smooth curve, it will approximate the required parabola.

(*b*) *By continuous motion.*

A right triangle HQD, Fig. 77, slides along DD', the directrix, the edge HQ being perpendicular to DD'. A string just long enough to reach from H to Q has one end fastened to the ruler at H; the other end is fastened at F, the focus of the parabola. The pencil point P presses against the string and ruler, keeping the string taut. As the triangle slides along DD' the pencil point P will trace the arc of a parabola, for $FP = QP$.

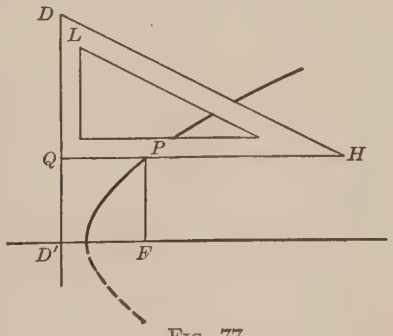


FIG. 77.

113. To Find the Equation of the Parabola. In order to get the simplest form of the equation, let F be the focus and DD' the directrix. Drop a perpendicular from F to DD' and choose this line as the x-axis, the direction N to F being positive. Choose as the origin O, the point midway

between N and F, and let $OF = p$. Then the coordinates of F are $(p, 0)$, and the equation of DD' is $x = -p$.

Now if $P(x, y)$ is any point on the locus, the distances MP and FP are by definition equal. But, $MP = x + p$, and

$$FP = \sqrt{(x - p)^2 + y^2};$$

therefore,

$$\sqrt{(x - p)^2 + y^2} = x + p. \tag{4}$$

Squaring and simplifying, we get

$$y^2 = 4px, \tag{5}$$

which is the equation of a parabola (Fig. 78) whose focus is at $(p, 0)$. Its principal axis is the x-axis; its vertex is at the

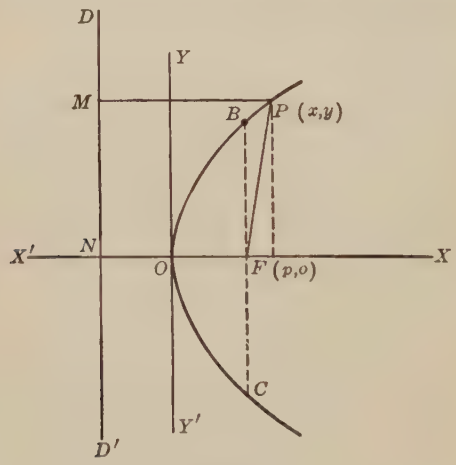


Fig. 78.

origin, O. The focal chord, BC, perpendicular to the axis is called the latus rectum. Its length, sometimes called the focal width, is equal to $4p$. This equation of the parabola is the basis of our study of the properties of this curve.

$y^2 = 4px$

EXERCISES

1. Draw the graph of each of the following equations. Give the coordinates of (*a*) focus, (*b*) vertex. Find the length of the latus rectum.

(*a*) $y^2 = 4x$.	(*e*) $x^2 = 3y$.	(*i*) $x^2 = 6y$.
(*b*) $y^2 = 8x$.	(*f*) $y^2 = -6x$.	(*j*) $y^2 = 6x$.
(*c*) $y^2 = 2x$.	(*g*) $y = x^2$.	(*k*) $x^2 = -6y$.
(*d*) $x^2 = 4y$.	(*h*) $x^2 = -4y$.	(*l*) $y^2 = 16x$.

2. The distance D feet through which a body falls in t seconds of time is given by the formula $D = 16t^2$. Plot the graph, using the horizontal axis as the time line and distance down as the positive direction of D.

3. How long will it take an iron ball to reach the earth if dropped from the top of (*a*) Bunker Hill monument, 221 feet high? (*b*) Washington Monument, 555 feet high? (*c*) Woolworth Building in New York City, 750 feet? (*d*) Eiffel Tower, 984 feet high? (*e*) an elevation equal to that of Mt. Everest, 29 000 feet high? (*f*) an airplane 6 miles high?

4. A bomb is dropped from an airplane 5280 feet high. Find its height and velocity 2 seconds later. 4 seconds later. 10 seconds later. When will it reach the ground? Velocity $= 32t$.

5. At a horizontal distance (x feet) from the middle of a certain cable in a suspension bridge, the height (y feet) above the lowest point is given by $y = 0.01x^2$. Plot the curve from $x = 0$ to $x = 30$.

6. An airship is traveling horizontally at a rate of 60 miles per hour when a bomb is dropped. Taking the starting point as the origin, find the equation of the path described by the bomb. Find the horizontal distance traversed by the bomb while falling 784 feet.

7. The velocity of an object falling from rest is given approximately by the formula $V = 8\sqrt{s}$. Plot the graph.

8. A parabolic reflector is 8 inches across and 10 inches deep. How far is the focus from the vertex?

9. After t seconds, the height s of a body thrown vertically upward with a velocity of 128 feet per second is $s = 128t - 16t^2$. Plot the graph. Find the maximum height reached, and the total time in the air.

10. A batted ball travels through the air subject to the following law: $x = 120t$, $y = 160t - 16t^2$, where t is the time in seconds. Calculate the coordinates (x, y) for each second from $t = 0$ to $t = 10$, and plot the graph.

11. The distance (s ft.) that a ball will roll down a certain incline in t seconds is approximately $s = 6t^2$. How far will it roll in 1 second? 2 seconds? 5 seconds?

12. The cross-section of the upper surface of a rotating fluid is parabolic, 5 inches deep and 100 inches across. Find the equation of the curve, taking the lowest point as (0, 0). Find y at the points where $x = 5$, $x = 10$, $x = 20$.

13. The suspension cable of a bridge is so loaded as to hang in the form of a parabola, PMQ. Its ends, which are 400 feet apart, are 50 feet above the lowest point. Find its simplest equation and also its height above the vertex at intervals of 20 feet.

Fig. 79.

14. The longitudinal section of a reflector of an automobile headlight is a parabola, 10 inches across and 8 inches deep. Obtain the equation of the curve, locate the focus, and construct the parabola (longitudinal section).

15. A parabolic sewer arch, AOB (Fig. 80), has a width of 12 feet and a height of 8 feet. Find the equation of such a cross-section as that drawn in Fig. 80.

16. A level footbridge of span 80 feet is supported by a parabolic suspension cable from points 19 feet above the floor. If the lowest point of the cable is 3 ft. above the floor of the bridge, find the height of the cable above the floor at intervals of 8 feet from the center.

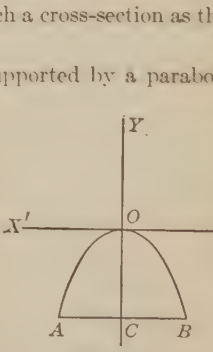


Fig. 80.

17. Write the equation of a parabola having its focus at (6, 0) and directrix $x = -6$.

18. One of the reflecting telescopes at Mt. Wilson has a parabolic mirror 60″ across. The distance from the vertex to the focus is 25 feet. Find the equation of a longitudinal section, the principal axis being the y-axis. How deep is the mirror at the center?

19. A roadway 36 feet wide is 8″ lower at the sides than along the middle. If the curve of the cross-section of the roadway is a parabola, find its equation. Find the drop, in inches, at intervals of 3 feet from the middle.

114. Another Mechanical Method. Engineers sometimes use the following method of constructing a drawing for a parabolic arch. If it is desired to construct a parabolic arch of width 40 feet and height 16 feet, a rectangle 20 by 16 is drawn (Fig. 81). The right-hand side of the rectangle

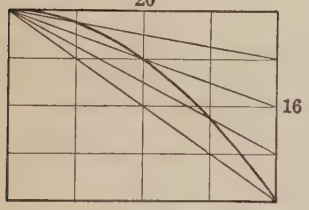


Fig. 81.

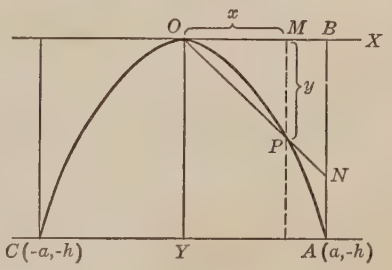


Fig. 82.

is divided into n equal parts, and the points of division are joined by straight lines to the upper left-hand vertex, which is also the vertex of the parabola. The upper (horizontal) side is also divided into n equal parts and ordinates are drawn at these points. The number n may be chosen to be any convenient integer. The intersection of the first ordinate and the first radiating line, the second ordinate and the second radiating line, etc., are points on the curve.

Proof. Take the axes OX and OY as in Fig. 82. Let $P(x, y)$ be any point on the curve. Then by construction BN and OM are equal parts of BA and OB respectively. Hence

$$\frac{BN}{BA} = \frac{OM}{OB}. \tag{6}$$

Now if $OM = x$, $OB = a$, $MP = y$, $BA = -h$, then

$$\frac{BN}{-h} = \frac{x}{a}, \quad \text{or} \quad BN = -\frac{hx}{a}. \tag{7}$$

Also from the similar triangles OMP and OBN

$$\frac{BN}{a} = \frac{y}{x}, \quad \text{or} \quad BN = \frac{ay}{x}. \tag{8}$$

Equating the values of BN from (7) and (8) and reducing,

$$x^2 = \frac{-a^2}{h} y. \tag{9}$$

This is the equation of a parabola. It passes through the points O, $C(-a, -h)$, and $A(a, -h)$, as required.

EXERCISES

1. By the method of § 114 construct a parabolic arch having a width of 80 feet and a height of 20 feet. Find its equation.

2. By the method of § 114 construct a parabolic arch having a width of 200 feet and a height of 40 feet. Find its equation.

3. The cable of a suspension bridge hangs in the form of a parabola. The width of the arch is 400 feet and the depth 50 feet. Construct the arch.

4. The width of a parabolic arch is 120 feet and the height is 36 feet. Construct this arch, using methods (*a*) and (*b*) § 112, and the method of § 114. Compare the three methods as to (*a*) time required for the construction, (*b*) accuracy.

5. If the weight of a suspension bridge is uniformly distributed over the length of the bridge, the cable takes the form of a parabola, nearly. The width between the suspension points in the Brooklyn bridge is 1500 feet and the vertex of the curve is 140 feet below the points of suspension. Assuming the curve to be a parabola, find (*a*) its equation, (*b*) the focal distance. Construct the curve by the method of § 114.

6. The reflecting telescope of the Detroit observatory, University of Michigan, has a diameter (arch span) of 37.5 inches. The distance from the vertex to the focus is 19.1 feet. Find the equation of the curve. Construct the curve by one of the mechanical methods.

CHAPTER VII

THEORY OF EQUATIONS

115. Algebra in Arithmetic Computation. Because of the very large use of the formula and graph in medicine, biology, sociology, chemistry, and education, the numerical side of mathematical work plays an important rôle. In physics, astronomy, agriculture, and engineering, a large amount of numerical computation through the use of logarithms and trigonometric functions is essential. Frequently other algebraic methods will be found useful in saving time and labor in computations. The value of the exponent (logarithm) has already been discussed. (See § 51.). Some formulas from elementary algebra will now be used.

116. Products and Roots. In multiplication, the most familiar formulas of elementary algebra are

(1) $(x + a)^2 = x^2 + 2ax + a^2$,
(2) $(x - a)^2 = x^2 - 2ax + a^2$,
(3) $(x + a)(x - a) = x^2 - a^2$,
(4) $(x + a)(x + b) = x^2 + (a + b)x + ab$.

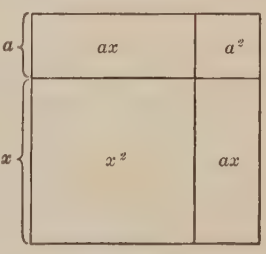


Fig. 83.

I. $(x \pm a)^2$ gives a simple method of squaring numbers.

Example. To square a number ending in 5, say 65.

$$\begin{aligned} 65^2 = (60 + 5)^2 &= 60^2 + 10 \cdot 60 + 25 \\ &= 60(60 + 10) + 25 \\ &= 60 \times 70 + 25 = 4200 + 25 \\ &= 4225. \end{aligned}$$

A short rule for squaring a number ending in 5 is as follows: multiply the ten's digit by one more than itself, and annex 25.

II. Use of $(x+a)(x-a)$ in obtaining products of numbers.

$$(x + a)(x - a) = x^2 - a^2.$$

EXAMPLE 1. Find the product of 14×16.

$$14 = 15 - 1, \qquad 16 = 15 + 1.$$
$$(15 + 1)(15 - 1) = 15^2 - 1^2 = 224.$$

EXAMPLE 2. Find the product of 14×18.

$$14 = 16 - 2, \qquad 18 = 16 + 2.$$
$$(16 - 2)(16 + 2) = 16^2 - 4 = 252.$$

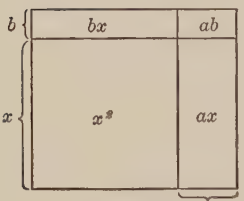


FIG. 84.

III. Use of $(x + a)(x + b)$, where $a + b = 10$.

EXAMPLE. Find the product of 62×68.

$$62 = 60 + 2, \qquad 68 = 60 + 8.$$
$$(60 + 2)(60 + 8) = 60^2 + 10 \times 60 + 16$$
$$= 60 \times 70 + 16$$
$$= 4216.$$

APPROXIMATE ROOTS. The student has used two methods for finding the square root of a number: (*a*) the method commonly taught in arithmetic, (*b*) logarithms. A very convenient method of approximating the square root of a number for the purpose of checking results is to factor the given number into two nearly equal factors and take the arithmetic mean of these factors as the square root of the number. Let it be required to approximate $\sqrt{30}$. $\sqrt{30} = \sqrt{6 \times 5}$. The arithmetic mean is $(6 + 5)/2 = 5.5$ Hence $\sqrt{30} = 5.5$ nearly. This process may be repeated by dividing 30 by 5.5, thus: $30 = 5.5 \times 5.454 \cdots$, and by averaging these factors again: $\sqrt{30} = 5.477 \cdots$, nearly.

EXERCISES

1. Expand each of the following expressions.

(*a*) $(x + a)^2$.	(*e*) $(100 + x)^2$.	(*i*) $(a + 2)(a - 2)$.
(*b*) $(x - a)^2$.	(*f*) $(50 + x)^2$.	(*j*) $(a - 1)(a + 1)$.
(*c*) $(x + 1)^2$.	(*g*) $(50 - x)^2$.	(*k*) $(a + 5)(a + 5)$.
(*d*) $(x + 2)^2$.	(*h*) $(100 - x)^2$.	(*l*) $(a + 4)(a + 6)$.

2. Find each of the following products.

(*a*) $(x + 7)(x + 3)$.	(*c*) $(15 + x)(15 - x)$.	(*e*) $(18 + 1)(18 - 1)$.
(*b*) $(x + 1)(x + 9)$.	(*d*) $(25 + x)(25 - x)$.	(*f*) $(16 + 2)(16 - 2)$.

3. Factor each of the following expressions.

(*a*) $x^2 - 4x + 4$.	(*f*) $4x^2 - 9$.	(*k*) $169 - 4$.
(*b*) $x^2 - 16x + 64$.	(*g*) $x^2 - 3x + 2$.	(*l*) $289 - 25$.
(*c*) $4x^2 - 4x + 1$.	(*h*) $x^2 - 8x + 15$.	(*m*) $81 - 1$.
(*d*) $x^2 - y^2$.	(*i*) $x^2 - 10x + 21$.	(*n*) $a^2 - 1/4$.
(*e*) $a^2 - 25$.	(*j*) $100 - 36x^2$.	(*o*) $a^2 - 1/81$.

4. Read the squares of each of the following numbers. The type is $(a + b)^2$, where $b = 5$.

$15^2 =$, $35^2 =$, $55^2 =$, $75^2 =$, $95^2 =$, $115^2 =$,
$25^2 =$, $45^2 =$, $65^2 =$, $85^2 =$, $105^2 =$, $125^2 =$.

5. Read each of the following products. The type form is $(x + a)(x + b)$ where $a + b = 10$.

(*a*) 62×68.	(*d*) 81×89.	(*g*) 18×12.	(*j*) 61×69.
(*b*) 73×77.	(*e*) 78×72.	(*h*) 23×27.	(*k*) 22×28.
(*c*) 34×36.	(*f*) 93×97.	(*i*) 34×36.	(*l*) 39×31.

6. Write the squares of the integral numbers from 12^2 to 25^2, inclusive. Time yourself.

7. Read each of the following products. The type is $(x+a)(x-a)$.

(*a*) 13×17.	(*d*) 19×17.	(*g*) 52×48.
(*b*) 26×24.	(*e*) 19×15.	(*h*) 43×37.
(*c*) 28×22.	(*f*) 36×34.	(*i*) 33×27.

8. Using the formula $(50 \pm a)^2$, time yourself on writing the squares of the numbers 40 to 60, inclusive.

9. Using the formula $(100 \pm a)^2$, time yourself on writing the squares of the numbers 90 to 110, inclusive.

10. Find the approximate square roots of 42, 132, 99, 240, 90, 6, 20, 56, 80, 109, 600, 72, 110, 30, 12, 156, 420, 240, 930.

117. Formation of Equations. It is sometimes necessary to form equations whose roots are given values.

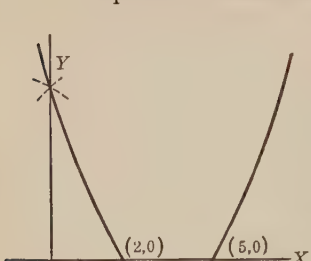


FIG. 85.

EXAMPLE 1. Let it be required to find the equation whose roots are 2 and 5.

From the definition of a root of an equation we have $x = 2$ or else $x = 5$. Transposing, we have $x - 2 = 0$ or else $x - 5 = 0$. Multiplying these together, we find

$$x^2 - 7x + 10 = 0,$$

which is true if either $x - 2 = 0$ or $x - 5 = 0$. The equation $y = x^2 - 7x + 10$ represents a parabola which crosses the x-axis at the points (2, 0) and (5, 0).

EXAMPLE 2. Let it be required to form the equation whose roots are 1, 3, and 4. Then

$$x = 1, \quad \text{or} \quad x = 3, \quad \text{or} \quad x = 4.$$

Transposing, we have $x - 1 = 0$, or $x - 3 = 0$, or $x - 4 = 0$. Multiplying, we find

$$x^3 - 8x^2 + 19x - 12 = 0.$$

The graph of the cubic equation $y = x^3 - 8x^2 + 19x - 12$ crosses the x-axis at the points (1, 0), (3, 0), and (4, 0).

118. Relation between Roots and Coefficients. It is required to form an equation whose roots are a and b. We have $x = a$ or else $x = b$. Transposing, we find $x - a = 0$ or else $x - b = 0$. Multiplying, we have

$$x^2 - (a + b)x + ab = 0, \tag{5}$$

which will be true if either $x - a = 0$ or $x - b = 0$.

Similarly, the equation whose roots are a, b, and c is

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0.$$

In general the coefficient of the next to the highest power of x is the sum of all the roots with their signs changed;

that of the third term is the sum of the products taken two at a time; that of the fourth term is the sum of the products taken three at a time with their signs changed; the last term is the product of all the roots with their signs changed. If the roots are all positive, the signs alternate, the sign of the second term being negative. If the roots are all negative, the signs are all positive.

119. Number of Roots. In the preceding paragraph we have seen that an equation having two roots is of the second degree; one having three roots is of the third degree.

An equation having n roots is of the nth degree, since the degree of the equation equals the number of binomial factors. Conversely, to each root a of the equation corresponds a factor $x - a$ by which the equation can be divided without a remainder. (See § 120.) Each time the equation is divided by a factor corresponding to a root, the degree of the equation as well as the number of roots is reduced by one. Since an equation of the first degree has but one root, an equation of the nth degree has only n roots.

The roots of an equation are not necessarily all different. The factors of $x^3 - x^2 - 8x + 12 = 0$ are $(x - 2)^2 \times (x - 3)$ and the roots are said to be 2, 2, and -3. This equation has two equal roots.

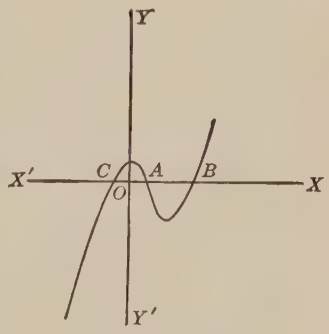


FIG. 86.

An equation may have no real roots. The equation $x^4 + 1 = 0$ cannot be satisfied by any real value of x, since every even power of a real number is a positive quantity. Such roots are called ***imaginary.***

Figure 86 shows the graph of $y = x^3 - 3x^2 + 2$. The intercepts on XX' are OA, OB, and OC. Hence the roots are all real and different. Figure 87 shows the graph of $y = x^3 - 3x^2 + 4$ in which two of the roots are equal. The graph is tangent to the x-axis at the point (2, 0). The roots are 2, 2, and -1. Figure 88 shows the graph of $y = x^3 - 3x^2 + 6$ in which two of the roots are imaginary. The curve intersects the x-axis but once.

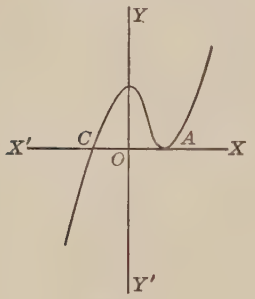


FIG. 87.

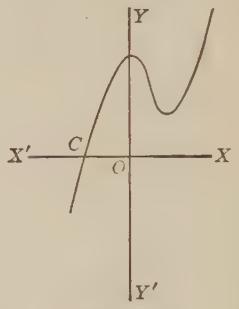


FIG. 88.

EXERCISES

1. Form the equation whose roots are:

(*a*) 1 and 5.
(*b*) 2 and 7.
(*c*) 3 and 8.
(*d*) 2, 4, and 5.
(*e*) 3, 5, and 6.
(*f*) 1, 2, and 4.
(*g*) 2 and -3.
(*h*) 3 and -3.
(*i*) 5 and -5.

2. Draw the graph of each of the following equations and read the values of the roots from the graph. Check the result.

(*a*) $y = x^3$.
(*b*) $y = x^3 - 4x^2$.
(*c*) $y = x(x - 3)^2$.
(*d*) $y = x^3 - 6x^2 + 11x - 6$.
(*e*) $y = x^3 - 9x^2 + 23x - 15$.
(*f*) $y = x^3 - 7x^2 + 10x$.

3. Draw the graph of each of the following functions.

(*a*) $x^3 - 6x^2 + 9x + 3$.
(*b*) $x^3 - 7x + 6$.

Discuss the nature of the roots of the corresponding equations.

4. By the formula find the roots of each of the following quadratic equations, and discuss their nature.

(*a*) $x^2 - 7x + 6 = 0$.
(*b*) $x^2 - 8x + 16 = 0$.
(*c*) $ax^2 + bx + c = 0$.
(*d*) $x^2 - 8x + k = 0$.

120. Remainder Theorem. For convenience, the general polynomial of the nth degree will be represented by the symbol $f(x)$, read (see § 4) "f function of x," or more briefly "f of x." We shall write

$$(6) \quad f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

We may state the following theorem regarding $f(x)$, it being understood that the letter r used below represents any given number.

Remainder Theorem. *If $f(x)$ is divided by $x - r$, the remainder is $f(r)$, where $f(r)$ indicates the value of $f(x)$ when r is substituted for x.*

This remainder must be a constant since it is of lower degree in x than the divisor, which is of the first degree. The quotient must be a polynomial in x, except when $n = 1$.

Proof. Let the quotient be represented by $Q(x)$ and the remainder by R; then since the dividend equals the divisor times the quotient plus the remainder, we have

$$(7) \quad f(x) = Q(x)(x - r) + R.$$

In this identity if we put $x = r$, we obtain:

$$(8) \quad f(r) = Q(r)(r - r) + R.$$

Since $r - r = 0$, we have

$$(9) \quad R = f(r).$$

The student should observe that R is independent of x.

Corollary. *If r is a root of the equation $f(x) = 0$, then $x - r$ is a factor of $f(x)$.*

For, by the theorem, the remainder obtained by dividing $f(x)$ by $x - r$ is $f(r)$ and by hypothesis $f(r) = 0$.

Note. This corollary is sometimes called the ***Factor Theorem.***

121. Synthetic Division. The work of dividing a polynomial by $x - r$ may be shortened by using the following abridged form, which is known as synthetic division.

Suppose it is desired to divide $x^3 - 6x^2 + 11x - 13$ by $x - 2$. By the ordinary method of long division, the process is as follows.

$$\begin{array}{l} x^3 - 6x^2 + 11x - 13 \mid x - 2 \\ x^3 - 2x^2 \qquad\qquad\qquad\qquad x^2 - 4x + 3 = \text{quotient.} \\ \hline\quad - 4x^2 + 11x \\ \quad - 4x^2 + 8x \\ \hline\qquad\quad + 3x - 13 \\ \qquad\quad 3x - 6 \\ \hline\qquad\qquad - 7 = \text{remainder.} \end{array}$$

The work may be shortened by using the coefficients only. If any power is wanting it should be supplied, the coefficient being zero. The several terms of the dividend need not be brought down. The process in its essentials is as follows.

$$\begin{array}{l} 1 - 6 + 11 - 13 \mid 1 - 2 \\ \quad - 2 + 8 - 6 \\ \hline 1 - 4 + 3 - 7 \end{array}$$

If the first term in the divisor is omitted since it no longer appears in the process, and the -2 is changed to $+2$, which is a root when $x - 2$ is a factor, the partial products in the second row will be added instead of subtracted. The work is written in the following form.

$$\begin{array}{l} 1 - 6 + 11 - 13 \mid 2 \\ \quad + 2 - 8 + 6 \\ \hline 1 - 4 + 3 - 7 \end{array}$$

The last term below the line is the remainder. The preceding terms are the coefficients in the quotient. In this problem the quotient is $x^2 - 4x + 3$, and the remainder is -7.

122. Removal of Exact Roots. When an exact root (r) has been found, the equation may be reduced to a lower degree by dividing by $x - r$.

Thus in the equation $x^4 - 2x^3 + 12x^2 - 37x + 26$, 2 is found to be a root. Dividing by $x - 2$, we may write

$$\begin{array}{r} 1 - 2 + 12 - 37 + 26 \quad \lfloor +2 \\ +2 + 0 + 24 - 26 \\ \hline 1 + 0 + 12 - 13 \end{array}$$

The quotient is $x^3 + 12x - 13$. By inspection, 1 is seen to be a root; hence $x - 1$ is a factor. Dividing, we have

$$\begin{array}{r} 1 + 0 + 12 - 13 \quad \lfloor +1 \\ +1 + 1 + 13 \\ \hline 1 + 1 + 13 \end{array}$$

The quotient is $x^2 + x + 13$. Setting this equal to zero, we have a quadratic equation whose roots are $(-1 \pm \sqrt{-51})/2$.

123. Transformation of an Equation. We can transform an equation into another whose roots are a given multiple of those of the first. Let the equation be

$$x^n + bx^{n-1} + cx^{n-2} + \cdots + tx^2 + ux + v = 0. \tag{10}$$

Put $x = y/m$; then whatever value x may have, y will have a value m times as great. Substituting, the equation becomes

$$\left(\frac{y}{m}\right)^n + b\left(\frac{y}{m}\right)^{n-1} + c\left(\frac{y}{m}\right)^{n-2} + \cdots + t\left(\frac{y}{m}\right)^2 + u\left(\frac{y}{m}\right) + v = 0. \tag{11}$$

Multiplying each term by m^n, we obtain

$$y^n + mby^{n-1} + m^2cy^{n-2} + \cdots + m^{n-2}ty^2 + m^{n-1}uy + m^nv = 0. \tag{12}$$

It will be readily seen that the desired transformation is effected by multiplying the second term by the given factor, the third term by its square, and so on.

EXERCISES

1. Obtain the quotient and remainder in each of the following exercises. Use synthetic division. Check the correctness of the remainder by the remainder theorem.

(*a*) $x^3 + 3x^2 + 5x - 4$ divided by $x - 2$.
(*b*) $x^3 + x^2 + 5x - 6$ divided by $x - 3$.
(*c*) $x^4 + 2x^3 + 3x^2 + 5x - 14$ divided by $x - 4$.
(*d*) $x^3 - 17$ divided by $x - 2$.
(*e*) $2x^3 - 3x^2 + 4x - 7$ divided by $x - 3$.

2. Determine whether the numbers placed in parentheses after each equation are roots of the equation.

(*a*) $x^3 - 6x^2 + 11x - 6 = 0$.	(2), (3), (1).
(*b*) $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$.	(1), (2), (4), (5).
(*c*) $x^4 - 7x^3 + 10x^2 + 7x - 11 = 0$.	(1), (− 1).
(*d*) $x^4 - 13x^2 + 36 = 0$.	(1), (− 1), (3), (− 3).
(*e*) $x^4 + 3x^3 - 15x^2 - 19x + 30 = 0$.	(1), (3), (− 2), (− 5).

3. Solve each of the following equations, having given that one root is the value of x set opposite.

(*a*) $x^3 + 4x^2 - 10x - 4 = 0$,	$x = 2$.
(*b*) $x^3 - 5x^2 + 11x - 15 = 0$,	$x = 3$.
(*c*) $x^3 - 4x^2 + 15x - 12 = 0$,	$x = 1$.

4. Transform each of the following equations into equations whose roots are the roots of the given equation multiplied by the number set opposite.

(*a*) $x^2 - 5x + 6 = 0$	by 4.
(*b*) $x^3 - 6x^2 + 11x - 6 = 0$	by 3.
(*c*) $x^3 - 5x^2 - 4x + 7 = 0$	by 5.
(*d*) $x^3 + 2x^2 + 3x - 7 = 0$	by 10.
(*e*) $x^3 + x^2 + 5x - 8 = 0$	by 10.

5. State a simple, easy method of obtaining an equation whose roots are the roots of a given equation multiplied by 10.

6. If $2x^2 - 5x + 7$ is represented by the symbol $f(x)$, find $f(1)$, $f(2)$, $f(5)$, $f(3)$.

7. Given $f(x) = x^3 - x^2 + 3x + 2$, calculate $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(10)$.

8. Determine whether the numbers placed in parentheses after each equation are roots of the equation.

(a) $x^4 + 2x^3 - 3x^2 + 5x - 4 = 0$	(1), (2).
(b) $x^3 + 7x^2 - 5x - 10 = 0$	(2), (5).
(c) $x^3 + 8x^2 + 12x - 24 = 0$	(1), (2), (4), (6).
(d) $x^5 + x^4 + x^3 + 6x^2 + 12x - 48 = 0$	(1), (2), (8).
(e) $x^4 + x^2 - 36 = 0$	(1), (2), (4).

124. The Number of Real Roots. In advanced texts on algebra there are given methods for determining the exact number of real roots of an equation. It will suffice to give here a simpler method due to Descartes (1596–1650), which gives an upper bound to the number of real roots.

125. Descartes' Rule of Signs. Any change in sign between two consecutive terms of $f(x)$ is called a *variation in sign*. Only non-zero terms need be considered. Thus the equation

$$f(x) = x^4 + 2x^3 + 4x^2 - 7x + 6 = 0$$

contains two variations (the signs are $+ + + - +$), one from $+$ to $-$, and the other from $-$ to $+$.

DESCARTES' RULE OF SIGNS. *An equation $f(x) = 0$ cannot have more real positive roots than there are variations of sign in $f(x)$, nor more real negative roots than there are variations of sign in $f(-x)$.*

EXAMPLE. In the equation

$$f(x) = x^3 + 4x^2 + x - 6 = 0,$$

the signs are $+ + + -$. Since there is only one variation, there is not more than one positive root. Substituting $-x$ in place of x, we find

$$f(-x) = -x^3 + 4x^2 - x - 6 = 0.$$

The signs are $- + - -$. Since there are only two variations, the equation $f(x) = 0$ cannot have more than two negative roots.

VERIFICATION. Suppose the successive terms of a complete equation have the signs $+ - + + + - - +$. There are four variations. Let us now introduce a positive root by multiplying by x minus this root. Writing only the signs, the work may be arranged as follows:

$$
\begin{array}{l}
+\ -\ +\ +\ +\ -\ -\ + \\
+\ - \\
\hline
+\ -\ +\ +\ +\ -\ -\ + \\
\quad -\ +\ -\ -\ -\ +\ +\ - \\
\hline
+\ -\ +\ \pm\ \pm\ -\ \mp\ +\ -
\end{array}
$$

The ambiguous sign is placed where the sign to be used must be determined by the relative numerical values of the coefficients. Counting the smallest number of variations that can occur, we find there are five. Thus in this case a positive root is introduced and at least one variation is added.

EXERCISES

1. Test the following equations by Descartes' Rule of Signs for the number of positive and also the number of negative roots.

(*a*) $x^4 + 2x^2 + 3x - 4 = 0.$
(*b*) $x^4 - 3x^3 + 5x^2 - 6x + 7 = 0.$
(*c*) $x^6 + x^4 + x^2 + 8 = 0.$
(*d*) $x^8 - x^6 + x^4 - x - 10 = 0.$
(*e*) $x^5 + x^3 - x^2 + x - 6 = 0.$

2. Prove that the equation $x^5 - 1 = 0$ has only one real root.

3. Prove that the equation $x^4 - 2x^3 + 5 = 0$ has at least two imaginary roots.

4. Prove that the equation $x^6 - 10 = 0$ has four imaginary roots.

126. Translation of the *y*-Axis. For the purpose of determining approximately an irrational root of an equation $f(x) = 0$, where $f(x)$ is a polynomial, a method is used which diminishes the roots of the equation by a constant amount, say a, then diminishes the roots of the resulting equation by a constant amount, say b, and so on. Let the method be considered from a graphical standpoint. This is accomplished by moving the y-axis along the x-axis in the positive direction. Each successive translation diminishes the roots of the equation by an amount equal to the distance moved.

EXAMPLE. Construct the graph of

$$y = x^2 - 5x + 5.$$

This graph is a parabola. One root lies between 1 and 2, and a second root between 3 and 4 (Fig. 89). To approximate the root between 1 and 2, the y-axis may be moved 1 unit to the right to the position $O'Y'$. This translation diminishes the root by 1. The next part of the root is found to lie between .3 and .4. Hence the y-axis is next carried .3 unit to the right to the position $O''Y''$. By continuing the process the root may be approximated to any desired number of decimal places.

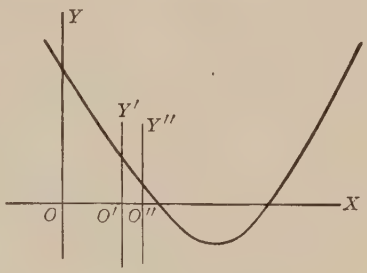


FIG. 89.

127. Horner's Method. In 1819, W. G. Horner of Bath, England, showed how to find successive digits in the roots of an equation, using the process of diminishing the roots. It applies only to equations in the standard polynomial form. The method of making these approximations will now be shown by a numerical illustration.

Let $f(x) = 0$ be the equation

$$x^3 + 2x^2 + 3x - 31 = 0. \tag{13}$$

From the graph, one root is seen to be between 2 and 3 (Fig. 90). To form the new equation whose roots are 2 less than the roots in the given equation, substitute $x' + 2$ for x, which gives

$$(x' + 2)^3 + 2(x' + 2)^2 + 3(x' + 2) - 31 = 0. \tag{14}$$

Now assume that this gives, when reduced,

$$x'^3 + Bx'^2 + Cx' + D = 0, \tag{15}$$

in which the coefficients B, C, and D can be found by expanding and combining terms in (14). The left-hand members of equations (14) and (15) are identical. Evidently then, if $x - 2$ is substituted for x' in (14) it will give the original equation. Likewise substituting $x - 2$ for x' in (15) gives the original equation. Substituting in (15), we have

(16) $(x - 2)^3 + B(x - 2)^2 + C(x - 2) + D = 0,$

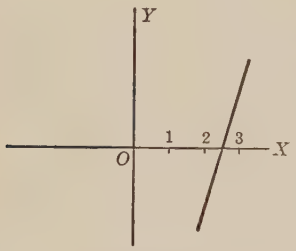


FIG. 90.

which is identical with the original equation. Now if we divide the left-hand member of (16) by $x - 2$ the remainder is D and the quotient is $(x - 2)^2 + B(x - 2) + C$. If this quotient is divided by $x - 2$, the remainder is C and the new quotient is $(x - 2) + B$. If the quotient of the preceding division is divided by $x - 2$, the remainder is B.

The actual numerical work is shown below.

$1 + 2 + 3 - 31$ $+ 2 + 8 + 22$	$\lfloor +2$	
$1 + 4 + 11 - 9, D$ $+ 2 + 12$		$D = -9$
$1 + 6 + 23 \ldots\ldots C$ $+ 2$		$C = 23$
$1 + 8 \ldots\ldots\ldots B$		$B = 8$

Hence the equation whose roots are 2 less than the roots of the original equation is

(17) $x^3 + 8x^2 + 23x - 9 = 0.$

By trial one root in equation (17) is found to be between .3 and .4; then (17) is transformed into an equation whose roots are less by .3; the work is as follows.

$$
\begin{array}{l|l}
1 + 8 \quad + 23 \quad - 9 & + .3 \\
\quad + \ .3 + \ 2.49 + 7.647 & \\
\hline
1 + 8.3 + 25.49 - 1.353 & \\
\quad + \ .3 + \ 2.58 & \\
\hline
1 + 8.6 + 28.07 & \\
\quad + \ .3 & \\
\hline
1 + 8.9 &
\end{array}
$$

Hence the second transformed equation, whose roots are .3 less than those of (17) and 2.3 less than those of (13), is

(18) $$x^3 + 8.9x^2 + 28.07x - 1.353 = 0$$

By trial the next part of the root is found to be between .04 and .05; hence one root of (13) is 2.34

This process can be continued until a root is approximated to any desired degree of accuracy.

SUGGESTIONS.

(*a*) To find the negative roots, substitute $(-x)$ for x in the equation. This changes the negative roots to positive roots without changing their numerical value.

(*b*) Instead of dividing by $x - a$ (tenths), the roots of both dividend and divisor may be multiplied by 10; the divisor then becomes $x - a$ (units), etc.

EXERCISES

1. Transform the equation $x^3 + 2x^2 + 5x - 61 = 0$ into an equation whose roots are less by 2.

2. Transform the equation $x^3 + 3x^2 - 2x - 49 = 0$ into one whose roots are less by 3.

Determine each of the following roots correct to three decimal places. Accompany each equation with its graph.

3. The root of $x^3 - 3x^2 + 6x - 11 = 0$ lying between 2 and 3.

4. The root of $x^3 + 2x^2 + 3x - 37 = 0$ lying between 2 and 3.

5. The root of $x^3 + 2x^2 + 5x - 17 = 0$ lying between 1 and 2.

6. The root of $x^3 + \quad 3x - 61 = 0$ lying between 3 and 4.

7. The root of $x^3 + \quad x - 73 = 0$ lying between 4 and 5.

Find to three or four decimal places the real roots of the following:

8. $x^3 - 4x^2 - 6x + 8 = 0$.

9. $x^3 + x - 1000 = 0$.

10. $x^3 + 10x^2 + 5x - 260 = 0$.

11. $x^3 - 3x^2 - 4x + 13 = 0$.

12. $x^3 - 5x^2 + 2x + 6 = 0$.

13. $x^3 - x^2 - 15x + 28 = 0$.

14. $x^3 - 2x^2 - 7x - 1 = 0$.

15. $x^3 + 2x^2 - x - 1 = 0$.

By solving a cubic equation, find the approximate value of:

16. $\sqrt[3]{3}$. **17.** $\sqrt[3]{21}$. **18.** $\sqrt[3]{65}$. **19.** $\sqrt[3]{38}$. **20.** $\sqrt[3]{35}$.

21. Find the fourth root of 291. Note that this is equivalent to solving the equation $x^4 - 291 = 0$.

22. The edge of a cube is 5 inches. By how much, correct to two decimal places, should each edge be increased in order to increase the volume to 142 cubic inches?

23. The dimensions of a rectangular box are 8 by 10 by 12 inches. By what amount, correct to two decimal places, should each edge be increased in order that the volume may be increased by 300 cubic inches?

24. The depth to which a sphere, three feet in diameter and having a specific gravity of 8/9, sinks in water is a root of the equation $2x^3 - 9x^2 + 24 = 0$. Find the smaller positive root. Why must the other roots be excluded?

25. In finding the maximum deflection of a 25 foot beam, loaded in a certain way, it can be shown that it is necessary to solve the equation $4x^3 - 150x^2 + 1500x - 3125 = 0$. Find one root.

26. A safe diameter (d in.) for the bolts in a certain steel shaft is a root of the equation $d^4 + 160d^2 - 85d - 536 = 0$. Find the diameter.

27. Archimedes proposed the problem "to cut a sphere by a plane in such a way that the two segments of the sphere have to each other a given ratio." If x is the distance of the plane from the center of the sphere whose radius is 5, then it can be shown that the problem leads to the cubic $x^3 - 300x + 2000(k - 1)/(k + 1) = 0$, where k is the ratio of the segments. Solve the equation if the plane cuts off one-third of the sphere, that is, $k = 2$.

28. An equation involved in computing the lateral stability of an aeroplane of given dimensions and having a steady horizontal motion is $x^4 + 9.31x^3 + 9.81x^2 + 10.15x - 0.161 = 0$. Find one value of x.

CHAPTER VIII

FIRST DEGREE EQUATIONS—USE OF DETERMINANTS

128. Historical Note. Problems leading to simple equations in one unknown are first found in an Egyptian papyrus called the Rhind mathematical papyrus. Eleven problems in this book might be regarded as equivalent to simple equations with one unknown, and are commonly referred to as the "hau-computations," since the unknown is called "*hau*," or "*heap*." Recent investigations indicate that *āḥā* would be more accurate than *hau*. The following is given to illustrate one of these problems.

Heap, its 2/3, its 1/2, its 1/7 its whole, it makes 33,

that is, in modern notation,

$$\frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x + x = 33.$$

Although the Egyptian method, with its unit fractions, was exceedingly clumsy, no advance seems to have been made until the time of Diophantus of Alexandria (third and fourth century A.D.). His method of solving equations was somewhat similar to that found in modern algebras, but he dealt only with positive quantities. About 700 A.D. the Hindus explained positive and negative numbers by means of assets and debts; but the Arabs, through whom the work of the Hindus reached Europe, were influenced also by Greek writings, and were unable to rise to the notion of a purely negative number. Hence that advantage was lost for a time.

129. Systems of Linear Equations. In many problems the relation between the unknowns may be expressed by two or more equations, called a *system of equations.* The simplest system consists of two linear equations, that is, two equations of the first degree with two unknowns. Such a system may be written in the form

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases} \tag{1}$$

By multiplying the first of these equations by b_2 and the second by b_1, then subtracting the two equations thus formed, we find

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \tag{2}$$

provided $a_1b_2 - a_2b_1 \neq 0$. Similarly, using a_2 and a_1 as multipliers, we find

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{3}$$

If the common denominator of the expressions on the right of the equality sign in (2) and (3) is not zero, these values of x and y give the solution of the system.

130. Determinants. The expression $a_1b_2 - a_2b_1$ which occurs in the denominator in both (2) and (3) can be expressed in a form which is called a ***determinant*** of the second order. It is usually written as a square array in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \tag{4}$$

and is called the determinant of the system (1). Similarly $c_1b_2 - c_2b_1$ and $a_1c_2 - a_2c_1$ may be written in the forms

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

The numbers a_1, a_2, b_1, b_2 are called elements of the deter-

minant (4). The elements a_1, b_2 (which lie along the diagonal passing through the upper left-hand corner of the determinant) form the principal diagonal. The elements a_2, b_1 form the secondary (or minor) diagonal.

To find the value of any determinant of the second order, subtract the product of the elements in the secondary diagonal from the product of the elements in the principal diagonal.

NOTE. The essential features of the method of solving equations by determinants were first published by Leibniz in 1693. The method was somewhat different from the modern form. No further advance was made until 1750 when Gabriel Cramer showed how to solve a system of n equations in n unknowns, and thus convinced mathematicians of the power of determinants. During the latter part of the 18th century, the investigations of Vandermonde and Gauss led to improvements in the theory, while Laplace and Lagrange made many important applications. The modern name determinant is due to Cauchy (1789–1857).

131. Solution by Determinants. The solution of the system of equations (1), § 129, may be written in the form

$$(5) \qquad x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

These forms are easily remembered as follows.

1. *The determinant in the denominator for both x and y is the same, being the determinant of the system.*

2. *The determinant in the numerator is obtained from the determinant of the system by replacing the column of coefficients of the unknown for which one is solving by the column of the absolute terms.*

By means of (5) the values of x and y may be obtained without the usual process of elimination.

EXAMPLE. Solve by determinants for x and y,

$$2x + 3y = 21,$$
$$3x - y = 4.$$

SOLUTION. Using formula (5), we obtain

$$x = \frac{\begin{vmatrix} 21 & 3 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{21(-1) - 3(4)}{2(-1) - 3(3)} = \frac{-21 - 12}{-2 - 9} = \frac{-33}{-11} = 3.$$

$$y = \frac{\begin{vmatrix} 2 & 21 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{2(4) - 3(21)}{2(-1) - 3(3)} = \frac{8 - 63}{-2 - 9} = \frac{-55}{-11} = 5.$$

Hence the solution is $x = 3$, $y = 5$.

CHECK. Substituting 3 for x, and 5 for y in the original equations, we obtain $6 + 15 = 21$ and $9 - 5 = 4$, which are true results.

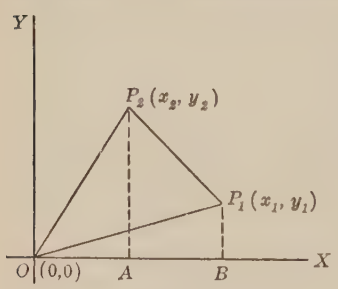


FIG. 91.

132. Area of a Triangle, One Vertex at the Origin. To find the area of the triangle OP_1P_2, Fig. 91, let AB be the projection on the x-axis of the line joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Then the area of the triangle OP_1P_2 = trapezoid BP_1P_2A + triangle OAP_2 − triangle OBP_1, which may be expressed in the form

$$\frac{(y_1 + y_2)(x_1 - x_2)}{2} + \frac{x_2y_2}{2} - \frac{x_1y_1}{2}. \tag{6}$$

Reducing, we find

$$\text{Area} = \frac{x_1y_2 - x_2y_1}{2} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{7}$$

EXAMPLE. Find the area of the triangle whose vertices are (0, 0), (5, 1), and (3, 7).

SOLUTION.

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 5 & 1 \\ 3 & 7 \end{vmatrix} = \tfrac{1}{2}(35 - 3) = 16.$$

EXERCISES

1. Evaluate each of the following determinants.

$(a)\ \begin{vmatrix} 3 & 2 \\ 5 & 9 \end{vmatrix}$, $(b)\ \begin{vmatrix} 8 & 2 \\ 4 & 7 \end{vmatrix}$, $(c)\ \begin{vmatrix} 9 & 7 \\ 1 & 3 \end{vmatrix}$, $(d)\ \begin{vmatrix} 5 & 9 \\ 4 & -2 \end{vmatrix}$, $(e)\ \begin{vmatrix} 5 & 6 \\ 1 & 3 \end{vmatrix}$.

2. Solve each of the following systems of equations by determinants and verify the solutions found.

$(a)\ \begin{cases} 5x + 2y = 20, \\ 3x + y = 11. \end{cases}$ $(d)\ \begin{cases} 3x + 5y = 22, \\ x + 2y = 8. \end{cases}$ $(g)\ \begin{cases} 7x + 2y = 17, \\ 3x - y = -2. \end{cases}$

$(b)\ \begin{cases} 4x + 7y = 19, \\ 5x + 3y = 18. \end{cases}$ $(e)\ \begin{cases} x + 2y = 7, \\ x - 3y = 2. \end{cases}$ $(h)\ \begin{cases} 4x + 3y = 17, \\ x - 2y = -4. \end{cases}$

$(c)\ \begin{cases} 6x + 2y = 14, \\ x + y = 5. \end{cases}$ $(f)\ \begin{cases} 3x + 5y = 30, \\ 5x + 3y = 34. \end{cases}$ $(i)\ \begin{cases} 2x - 5y = 5, \\ 5x - 2y = -5. \end{cases}$

3. Find the area of each of the following triangles, the coordinates of the vertices being:

(a) (0, 0), (6, 2), and (1, 7). (d) (8, 9), (9, 3), and (0, 0).
(b) (0, 0), (7, 3), and (2, 8). (e) (3, 4), (0, 0), and (9, 12).
(c) (0, 0), (5, 3), and (7, 3). (f) (0, 0), (4, 5), and (1, 9).

4. Express, in the notation of determinants, the condition that the two roots of the equation $ax^2 + bx + c = 0$ shall be equal. The condition is $b^2 - 4ac = 0$.

5. Solve the following fifteen systems of equations, using (a) determinants, (b) addition and subtraction, (c) graphs. Record the exact time required for each method.

$(a)\ \begin{cases} 5x + 3y = 11, \\ 3x + 2y = 7. \end{cases}$ $(f)\ \begin{cases} 4x + 5y = 23, \\ x + 2y = 8. \end{cases}$ $(k)\ \begin{cases} 4x + 2y = 10, \\ 3x - 5y = 1. \end{cases}$

$(b)\ \begin{cases} 7x + 2y = 17, \\ x + y = 6, \end{cases}$ $(g)\ \begin{cases} 3x - 5y = -14, \\ x + 3y = 14. \end{cases}$ $(l)\ \begin{cases} 6x + 5y = 4, \\ 2x - 3y = -1. \end{cases}$

$(c)\ \begin{cases} 3x + 7y = 29, \\ 2x + y = 12. \end{cases}$ $(h)\ \begin{cases} x - 5y = -11, \\ 5x + y = 23. \end{cases}$ $(m)\ \begin{cases} 5x + y = -13, \\ 3x - 2y = 0. \end{cases}$

$(d)\ \begin{cases} x + 2y = 9, \\ 2x + y = 6. \end{cases}$ $(i)\ \begin{cases} 3x - y = 11, \\ 5x + 2y = 22. \end{cases}$ $(n)\ \begin{cases} 7x + 9y = 25, \\ 2x - 4y = -6. \end{cases}$

$(e)\ \begin{cases} 3x + 5y = 21, \\ 5x + 3y = 19. \end{cases}$ $(j)\ \begin{cases} 3x + 2y = -12, \\ x - 5y = 13. \end{cases}$ $(o)\ \begin{cases} 2x + 3y = -12, \\ 5x - 2y = -11. \end{cases}$

133. Three Equations with Three Unknowns. The general form of a system of three equations in three unknowns x, y, and z is

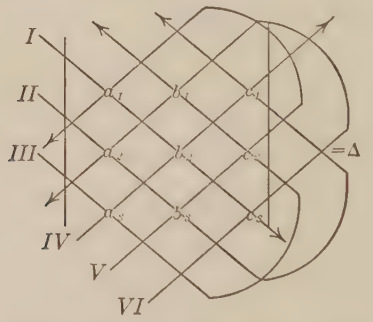
$$(8)\qquad \begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

This system may be solved for x, y, and z by elimination but the process is long. We shall give merely the results.

$$(9)\qquad \begin{cases} x = \dfrac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \\ y = \dfrac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \\ z = \dfrac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \end{cases}$$

The denominator common to these fractions may be

(10)



expressed as *a determinant of the third order*. It contains nine elements which may be arranged in the square array (10).

By inspection of the terms in the expanded determinant (10) it will be seen that the three positive terms are the

products of elements lying on the arrows that start downward and to the right, that is, I, II, and III. The three negative terms lie on the arrows ranging upward and to the right, that is, IV, V, and VI. Δ is called the determinant of the system. Careful inspection of (9) will show that the numerator of the x-value differs from the denominator only in the substitution of d_1, d_2, d_3 for a_1, a_2, a_3, respectively. Similarly the numerator of the y-value differs from the corresponding denominator only in the substitution of d_1, d_2, d_3 for b_1, b_2, b_3, respectively. Hence the values of the three unknowns may be written down as follows:

$$(11)\quad x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

These expressions constitute a formula for solving the system, provided the denominator is different from zero.

134. Development by Minors. If in a determinant one row and one column containing a given element, say b_k, are erased, the remaining elements form a determinant called the minor of b_k.

In the determinant

$$(12)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the minor of a_2 is

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix},$$

since the row and column erased both contain a_2. It will

be seen that the above determinant may be written in the expanded form

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1), \tag{13}$$

or

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \tag{14}$$

As thus written the determinant is said to be developed according to the minors of its first column. In like manner the above determinant may be developed according to the elements of any given row or column.

135. Properties of Determinants. The following theorems contain some of the most important properties of determinants.

I. *If the rows of a determinant Δ are replaced by the corresponding columns, the expansion is unchanged.*

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{15}$$

II. *If two rows (or columns) of a determinant Δ are interchanged, the sign of the determinant is changed.*

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}. \tag{16}$$

It will be seen that the interchanging of two columns will simply interchange two letters in each term of the expansion. It will change the signs of all the terms. Similarly, an interchange of two rows will change the signs of all of the terms.

III. *If a determinant Δ has two rows (or columns) identical, its value is zero.* If we interchange two rows, we obtain, by II, $-\Delta$. Since the interchange of two identical rows or columns would leave the determinant unaltered, we have $\Delta = -\Delta$, that is, $2\Delta = 0$, or $\Delta = 0$.

IV. *If all the elements of a row (or column) of a determinant Δ are multiplied by the same number m, the determinant is multiplied by m.* Since each term of the expansion contains one and only one element from any column, each term will be multiplied by m.

EXERCISES

Evaluate each of the following determinants.

1. $\begin{vmatrix} 1 & 2 & 7 \\ 2 & 1 & 6 \\ 3 & 2 & 4 \end{vmatrix}$.

2. $\begin{vmatrix} 3 & 4 & 1 \\ 5 & 1 & 6 \\ 0 & 2 & 3 \end{vmatrix}$.

3. $\begin{vmatrix} 4 & 1 & 1 \\ 3 & 5 & 4 \\ 6 & 0 & 2 \end{vmatrix}$.

4. $\begin{vmatrix} 17 & 0 & 0 \\ 1 & 5 & 6 \\ 1 & 3 & 7 \end{vmatrix}$.

5. $\begin{vmatrix} 9 & 2 & 0 \\ 3 & 7 & 0 \\ 5 & 2 & 1 \end{vmatrix}$.

6. $\begin{vmatrix} x_1 & 0 & 0 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

7. $\begin{vmatrix} 9 & 2 & 3 \\ 12 & 3 & 4 \\ 15 & 1 & 2 \end{vmatrix}$.

8. $\begin{vmatrix} 4 & 2 & 4 \\ 6 & 3 & 2 \\ 10 & 5 & 1 \end{vmatrix}$.

9. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & x-y & 0 \\ 0 & 0 & x+y \end{vmatrix}$.

Solve each of the following systems by determinants.

10. $\begin{cases} x + 2y + z = 8, \\ 3x + y + 2z = 11, \\ 5x + 2y + z = 12. \end{cases}$

11. $\begin{cases} x + y = 4, \\ y + z = 7, \\ x + z = 5. \end{cases}$

12. $\begin{cases} 3x + y + z = 13, \\ x + 2y + z = 12, \\ x + 3y + 2z = 19. \end{cases}$

13. $\begin{cases} 3x + y + z = 8, \\ 10x - y - 2z = 3, \\ 5x - 2y - 3z = -7. \end{cases}$

14. $\begin{cases} 7x + 3y + z = 17, \\ 2x + 4y - 2z = 6, \\ x + y + z = 9. \end{cases}$

15. $\begin{cases} 5x + y + z = 12, \\ 2x - 3y - 2z = -14, \\ x - y - 2z = -11. \end{cases}$

16. $\begin{cases} 3x + y + z = 6, \\ 5x + 2y - z = 8, \\ x + 3y + 2z = 9. \end{cases}$

17. $\begin{cases} 5x + 4y - 6z = 7, \\ x - y - z = -7, \\ 2x - 3y + 5z = 2. \end{cases}$

136. Area of a Triangle, General Case. Let the vertices of the triangle be $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and draw the ordinates M_1P_1, M_2P_2, and M_3P_3. Then the triangle

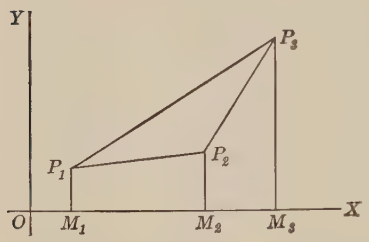


FIG. 92.

$$
\begin{aligned}
(17)\quad P_1P_2P_3 &= M_1P_1P_3M_3 \\
&\quad - M_1P_1P_2M_2 - M_2P_2P_3M_3 \\
&= \tfrac{1}{2}[M_1M_3(M_1P_1 + M_3P_3) \\
&\quad - M_1M_2(M_1P_1 + M_2P_2) \\
&\quad - M_2M_3(M_2P_2 + M_3P_3)] \\
&= \tfrac{1}{2}[(x_3 - x_1)(y_1 + y_3) \\
&\quad - (x_2 - x_1)(y_1 + y_2) \\
&\quad - (x_3 - x_2)(y_2 + y_3)].
\end{aligned}
$$

Expanding and collecting, we obtain

$$(18)\quad \text{Area} = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

Formula (18) may appear difficult to remember and apply. It may be simplified and remembered easily by expressing the result in the determinant form

$$(19)\qquad A = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

EXAMPLE. Find the area of the triangle of which the vertices are (2, 1), (7, 2) and (5, 8).

$$\text{Area} = \tfrac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 7 & 2 & 1 \\ 5 & 8 & 1 \end{vmatrix} = \tfrac{1}{2}[4 + 56 + 5 - 10 - 7 - 16] = 16.$$

NOTE. If in computing the area by formula (19) the vertices of a triangle are so numbered that in passing around the perimeter in that order the area is continually on the left the result is positive. If the area is on the right the result is negative.

137. Area of a Polygon. The area of any polygon can be found by breaking it up into triangles, and computing the areas of each of these triangles.

138. Determinant Form of the Two-point Equation. The student should notice that in formula (19) if one point (x, y) moves so as always to be on the line joining the other two points, the area is zero. Hence the equation of a straight line through the points (x_1, y_1) and (x_2, y_2) may be written in the easily remembered form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{20}$$

This equation, when the determinant is expanded, is identical with the two-point form of the equation of a straight line when cleared of fractions. See § 94.

EXERCISES

1. Find the area of the triangles whose vertices are

(*a*) (2, 3), (8, 4) and (6, 12).
(*b*) (3, 0), (7, 11) and (1, 9).
(*c*) (0, 5), (6, 1) and (9, 5).

2. Find the area of the triangle whose vertices are

(*a*) (0, 0), (8, 3) and (6, 7).
(*b*) (2, 0), (9, 4) and (7, 5).
(*c*) (0, 3), (7, 1) and (8, 5).
(*d*) (4, 3), (0, 0) and (9, 8).
(*e*) (2, 3), (5, 8) and (6, 7).
(*f*) (4, 5), (6, 2) and (8, 7).

3. Find the area of the quadrilateral whose vertices are

(*a*) (1, 2), (5, 3), (7, 6) and (2, 8).
(*b*) (1, 1), (6, 2), (8, 5) and (0, 7).

4. Find the equation of a straight line through the points

(*a*) (1, 1) and (5, 8).
(*b*) (2, 3) and (6, 10).
(*c*) (3, 0) and (7, 6).
(*d*) (4, 0) and (0, 8).
(*e*) (6, 0) and (3, 11).
(*f*) (4, 3) and (5, 6).

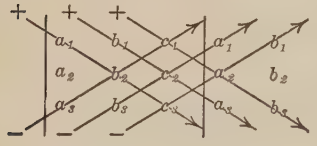
5. Prove that the three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ lie on a straight line if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

6. Determine whether the following sets of points lie on a straight line.

SUGGESTION. The determinant expressing the area must equal zero if the points are on a straight line.

(*a*) (0, 0), (1, 1) and (6, 6).
(*b*) (2, 1), (3, 7) and (4, 13).
(*c*) (− 2, − 1), (0, 1) and (5, 6).
(*d*) (2, 3), (− 4, − 6) and (8, 12).
(*e*) (− 2, − 3), (1, 1) and (7, 7).
(*f*) (*a*, *b*), (3*a*, 2*b*) and (− *a*, 0).



7. Write the determinant (10), § 133, with the first two columns repeated at the right. From the sum of the products of the elements on the diagonals marked +, subtract the sum of the products of the elements on the diagonals marked −. Compare the result of this expansion with that of § 133.

8. In a triangle whose vertices are (5, 3), (3, − 4), and (− 5, 6), lines are drawn joining the middle points of the sides. Prove that the area of the first triangle is four times that of the second.

9. Find the area of the trapezoid whose vertices are (0, 0), (*a*, 0), (*b*, *c*) and (*d*, *c*). Show that this area is the product of the altitude by one-half the sum of the parallel sides.

10. Find the area of the triangular field *ABO* (Fig. 93) if *OC* = 22 rods; *CD* = 16 rods; *DA* = 14 rods; *CB* = 32 rods.

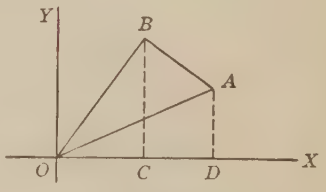


FIG. 93.

11. A surveyor finds that the corners of a four-sided field are situated with respect to a north and south road, and an east and west road, as follows: *A* = (20, 8), *B* = (48, 15), *C* = (32, 50), and *D* = (10, 40). Distances are in rods. Make a map of the field and compute the number of acres in it.

12. A surveyor finds that the corners of a four-sided field are situated with respect to a north and south line and an east and west line as follows: *A* = (6, 4), *B* = (32, 18), *C* = (28, 40), *D* = (7, 36). The distances are measured in rods. Make a drawing of the field and compute its value at $240 per acre.

CHAPTER IX

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

139. Notation. Let the coordinates of a given point on a curve be represented by (x, y) and that of a second point by $(x + \Delta x, y + \Delta y)$. Then Δx is the difference of the values of x, and Δy is the difference of the values of y. This is also expressed by saying that Δx is the *increment* of x, and Δy is the *increment* of y. The quotient $\Delta y/\Delta x$ is the slope of the line joining the points.

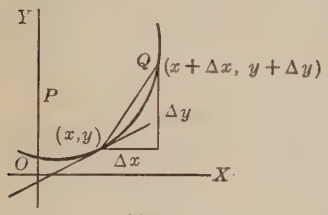


FIG. 94.

NOTE. Δ (the Greek letter delta) does not represent a number, but is used with a variable to denote the difference of two values of the variable.

140. The Slope of the Tangent. To find a general method for obtaining the slope of a tangent to a curve, as $y = x^2$, we proceed as follows.

At some point P on the curve, let the abscissa be x; then the ordinate is $y = x^2$. At a second point Q, the abscissa is $x + \Delta x$ and the ordinate $y + \Delta y = (x + \Delta x)^2$. Hence we have the two equations

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2, \tag{1}$$

$$y = x^2. \tag{2}$$

Subtracting, we obtain

$$\Delta y = 2x\Delta x + (\Delta x)^2. \tag{3}$$

Dividing by Δx, we get the slope of the line PQ:

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x. \tag{4}$$

Now if the point Q approaches along the curve closer and closer to P, the secant line PQ approaches the position of the tangent at P. This in fact may be taken as the

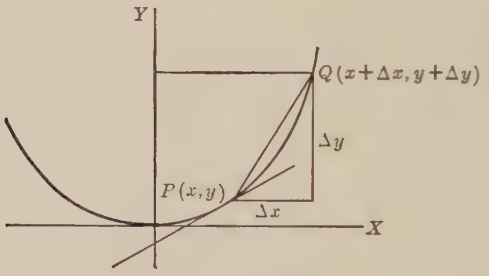


FIG. 95.

definition of a tangent. But the abscissa of Q minus the abscissa of P is Δx. Hence as Δx approaches zero as a limit, the slope of the secant approaches the *slope of the tangent*. The limit has the value $2x$ exactly.

The limit of $\Delta y/\Delta x$, when Δx approaches 0, written $\Delta x \to 0$, is usually expressed by the notation dy/dx and is called the ***derivative*** of y with respect to x. Hence if $y = x^2$,

$$\frac{dy}{dx} = 2x, \tag{5}$$

and we say that $2x$ is the derivative of x^2. Observe that we get the final value dy/dx, not by putting $\Delta x = 0$ (which would make $\Delta y/\Delta x$ meaningless), but by seeing what limit $\Delta y/\Delta x$ approaches, as Δx approaches zero as a limit.

NOTE. About 1665, while investigating problems of this nature, Sir Isaac Newton discovered the principles of the Calculus, one of the most beautiful as well as one of the most powerful tools in mathematics.

141. Interpretation of $\Delta y/\Delta x$ and dy/dx. If x is the abscissa and y the ordinate of any point on a curve, then Δy is the rise in a short horizontal run Δx, and $\Delta y/\Delta x$ is the average slope of the curve in the interval Δx. The limit of the average slope is dy/dx, which is defined as the slope at the point P.

If y denotes the distance traveled and x the time taken, then Δy denotes the additional distance traveled in the additional time Δx, and the *average speed* during the time Δx is

$$\frac{\Delta y}{\Delta x} = \frac{\text{distance}}{\text{time}}. \tag{6}$$

Hence the limit of the average speed, which is called the instantaneous speed, is

$$\frac{dy}{dx} = \text{instantaneous speed}. \tag{7}$$

Again, if y denotes the volume of any quantity which varies as x, the temperature, and Δy is the increase in y corresponding to the increase Δx in x, then $\Delta y/\Delta x$ is the average rate of increase in cubic units per degree, and dy/dx is the limit of the average rate, or the instantaneous rate.

142. Derivative of the General Quadratic Function. Let $y = ax^2 + bx + c$ be any quadratic function. Let x increase from the value x to $x + \Delta x$; then y will take on the corresponding increment Δy. Then

$$y = ax^2 + bx + c, \tag{8}$$

and, by definition of $y + \Delta y$,

$$y + \Delta y = a(x + \Delta x)^2 + b(x + \Delta x) + c. \tag{9}$$

Expanding, we have

$$y + \Delta y = ax^2 + 2ax\Delta x + a(\Delta x)^2 + bx + b\Delta x + c; \tag{10}$$

but, from (8),

$$y = ax^2 + bx + c. \tag{11}$$

Subtracting, we obtain

$$\Delta y = 2ax\Delta x + a(\Delta x)^2 + b\Delta x. \tag{12}$$

Dividing by Δx, we find

$$\frac{\Delta y}{\Delta x} = 2ax + a\Delta x + b. \tag{13}$$

When $\Delta x \to 0$ as a limit, we have $dy/dx = 2ax + b$.

STATEMENT. *The instantaneous rate of change of y with respect to x is represented by the slope of the tangent at the point.*

Hence the slope of the tangent to the curve

$$y = ax^2 + bx + c \tag{14}$$

at (x_1, y_1) is

$$\frac{dy}{dx} = 2ax_1 + b. \tag{15}$$

Similarly, the slope of the tangent to

$$y = ax^3 + bx^2 + cx + d \tag{16}$$

at (x_1, y_1) is

$$\frac{dy}{dx} = 3ax_1^2 + 2bx_1 + c, \tag{17}$$

and the slope of the tangent to

$$y = ax^n + bx^{n-1} \cdots \tag{18}$$

at (x_1, y_1) is

$$\frac{dy}{dx} = nax_1^{n-1} + b(n-1)x_1^{n-2} \cdots. \tag{19}$$

STATEMENT. *The derivative of any positive integral power of x equals the exponent of the given power times the next lower integral power of x.*

The process of calculating a derivative is called ***differentiation.***

143. Summary.

(1) *The derivative of a constant is zero.* If $y = c$, $dy/dx = 0$.

(2) *The derivative of a variable having a constant integral exponent is the product of the exponent times the variable with its exponent diminished by one.*

(3) *The derivative of the product of a constant and a variable is the constant multiplied by the derivative of the variable.* Thus the derivative of ax is a; of ax^3 is $3ax^2$; of ax^n is nax^{n-1} all with respect to x.

144. Formulas for Differentiation.

I. If $y = c$ (a constant),	then	$dy/dx = 0$.
II. If $y = x$,	then	$dy/dx = 1$.
III. If $y = x^n$,	then	$dy/dx = nx^{n-1}$.
IV. If $y = cx^n$,	then	$dy/dx = cnx^{n-1}$.

145. An Increasing or Decreasing Function. Frequently it is necessary to determine whether a given function $y = f(x)$ is increasing or decreasing with respect to x itself. To be definite, let x always increase. Then the graph

of the function is rising at any point where its slope is positive, and falling where its slope is negative. That is, the ordinate y is increasing where dy/dx is positive, and is decreasing where dy/dx is negative.

Consider the curve $y = x^2 - 5x + 4$. Then $dy/dx = 2x - 5$. At the point on the curve where $x = 0$, $y = 4$, and the slope $dy/dx = -5$; at $x = 1$, $y = 0$, and the slope $= -3$; at $x = 2\frac{1}{2}$, $y = -2\frac{1}{4}$, and the slope $= 0$; at $x = 4$, $y = 0$, and the slope $= 3$, etc.

x	y	dy/dx
0	4	-5
1	0	-3
$2\frac{1}{2}$	$-2\frac{1}{4}$	0
4	0	3

Since the slope is negative to the left, and positive to the right of $x = 2\frac{1}{2}$, the curve is falling for all values of x less than $2\frac{1}{2}$ and rising for values of x greater than $2\frac{1}{2}$.

146. Maxima and Minima Determined from the Slope. It has already been shown that maximum or minimum values may be found approximately from the graph. If in Fig. 96 a tangent line is drawn to the curve at the maximum point B or the minimum point C, it is parallel to or coincides with the x-axis, and consequently its slope is zero.

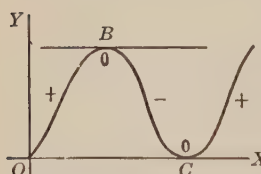


Fig. 96.

At any maximum or minimum point on a smooth curve the slope is zero; that is, $dy/dx = 0$. If for increasing values of x the slope changes from positive to negative, the curve has a maximum. If for increasing values of x the slope changes from negative to positive, the curve has a minimum.

Example. Find the slope of the curve $y = x^3 - 12x$, and the coordinates of the maximum and minimum points.

Solution.

$$y = x^3 - 12x.$$

Then the slope is

$$\frac{dy}{dx} = 3x^2 - 12 = 3(x^2 - 4).$$

Since for a maximum or minimum $dy/dx = 0$, we have $3(x^2 - 4) = 0$. Hence the curve may have a maximum or minimum at the points $x = \pm 2$. By substituting these values of x in the original equation it is found that when $x = -2$, $y = 16$, when $x = 2$, $y = -16$. Now if $x < -2$, say -2.1, $x^2 - 4$ is positive; if x is slightly greater than -2, say -1.9, $x^2 - 4$ is negative. Hence for $x = -2$, y is a maximum and has the value 16; similarly if $x = 2$, y is a minimum and has the value -16.

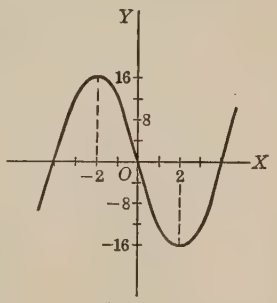


Fig. 97.

147. Historical Note. Sir Isaac Newton, the world's greatest mathematician and physicist, was born in England, December 25, 1642 (old style, January 5, 1643). He was educated at Trinity College, Cambridge, where his genius for mathematics showed itself even before his graduation in 1665. Within a year he had invented the calculus. He discovered many laws of physics, including the law of gravitation; and his writings fill many volumes. The *Principia*, his masterpiece, was published in 1687. His career as a student and teacher was a brilliant one, as may be gleaned from the following tributes to him.

"Nature and Nature's laws lay hid in night;
God said, 'Let Newton be!' and all was light."—Pope.

"There, Priest of Nature! dost thou shine.
Newton, a king among the kings divine."—Southey.

"Taking mathematics from the beginning of the world to the time when Newton lived, what he had done was much the better half."—Leibniz.

Of himself, Newton spoke modestly as follows:

"I do not know what I may seem to the world, but as to myself, I seem to have been only a boy playing on the seashore, and diverting myself now and then finding a smoother

pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me."

Ball's *History of Mathematics* gives an excellent account of Newton's life and works. Read it for the sake of the interest and inspiration it will give you.

EXERCISES

1. Find the value of dy/dx (slope) by using the full process in each of the following cases.

(*a*) $y = x^2 + 2$. (*d*) $y = x^2 - 6x$. (*g*) $y = x^3 + 4$.
(*b*) $y = x^2 + 2x$. (*e*) $y = x^2 + 3x - 7$. (*h*) $y = x^3 + x^2$.
(*c*) $y = x^2 + 3x$. (*f*) $y = x^2 + 2x + 5$. (*i*) $y = x^3 - 5x + 6$.

2. Along the arch of a bridge the height (y feet) above the water at any horizontal distance (x feet) from the center is given by the equation $y = 50 - .01x^2$. Find the slope of the tangent at $x = 0$, $x = 10$, $x = 50$.

3. State the derivatives of each of the following functions.

(*a*) $y = x^2 + 5x + 7$. (*g*) $y = x^3 - 3x^2 + 5x + 10$.
(*b*) $y = x^3$. (*h*) $y = 3 - x - x^2 + x^3 - x^4$.
(*c*) $y = x^3 + 6x^2 + 7x - 13$. (*i*) $y = .4x$.
(*d*) $y = x^2 + 11x - 5$. (*j*) $y = .05x^2$.
(*e*) $y = x^2 + 3x + 7$. (*k*) $y = -.3x^2$.
(*f*) $y = x^4 + x^3 - x^2 - x + 2$. (*l*) $y = \frac{1}{6}x^7$.

4. Find the numerical value of the slope of the tangent to each of the following curves when the value of x is the number set opposite.

(*a*) $y = x^2 + 6x + 5$, $x = -3$. (*d*) $y = x^3 - 3x - 1$, $x = 3$.
(*b*) $y = x^2 - 4x + 3$, $x = 2$. (*e*) $y = x^3 + x + 1$, $x = 4$.
(*c*) $y = x^2 + 3x + 1$, $x = 1$. (*f*) $y = x^2 + 2x + 2$, $x = \frac{1}{2}$.

5. Find the coordinates of the maximum or minimum point of each of the following curves. Compare with the results obtained from the graph of the equation.

(*a*) $y = x^2 - 6x + 11$.

SOLUTION. $dy/dx = 2x - 6$. Since at a maximum or minimum point the slope of the tangent is zero, put $2x - 6 = 0$.

Hence $x = 3$, the abscissa of a point which may be a maximum or minimum. When $x < 3$, dy/dx is negative; when $x > 3$, dy/dx is positive. Therefore, the point is a minimum. Substituting 3 for x in

the equation, $y = 9 - 18 + 11 = 2$, the ordinate of the minimum point. Hence the coordinates of the minimum point are (3, 2). The curve has no maximum.

(*b*) $y = x^2 - 8x + 10$.
(*c*) $y = x^2 + 2x + 2$.
(*d*) $y = 3 - 2x + x^2$.
(*e*) $y = x^2 - 18x$.
(*f*) $y = 5 - x - x^2$.
(*g*) $y = x^2 + 4x - 12$.
(*h*) $y = 6 - 4x + x^2$.
(*i*) $y = x^2 - 12x$.
(*j*) $y = 2x^2 + 8x - 6$.
(*k*) $y = x^3 + 6x^2 + 9x + 12$.
(*l*) $y = ax^2 + bx + c$.
(*m*) $y = x^3 - 9x + 4$.

6. Divide 10 into two such parts that their product is a maximum.

HINT. Let x be one part. Then $10 - x$ will be the other part, and the product is $x(10 - x)$.

7. Divide a into two such parts that their product is a maximum.

8. Divide 10 into two such parts that the sum of their squares is a minimum.

9. Divide a into two such parts that the sum of their squares is a minimum.

10. What number added to its reciprocal gives the least possible sum?

11. Divide the number 8 into two such parts that the product of one part by the cube of the other may be as large as possible.

12. Divide the number a into two parts such that the product of one part by the cube of the other may be as large as possible.

13. Divide the number 10 into two parts such that the square of one part multiplied by the cube of the other shall be a maximum.

14. Find the number that exceeds its square by the greatest possible quantity.

15. Find the dimensions of the greatest rectangular area that can be enclosed by 100 rods of fence.

16. A piece of tin 30 inches square has a small square cut out of each corner. The projecting parts are then folded up so as to form an open-top box of maximum volume. Find the dimensions of the box.

HINT. $V = x(30 - 2x)^2$.

17. A rectangle is to have a perimeter of 40 in. For what dimensions will its area be greatest?

18. A rectangular sheet of tin 20 feet long and 12 inches wide is to be made into a gutter by turning strips up vertically along the two

sides. How deep should the gutter be to secure the greatest carrying capacity, that is, the greatest sectional area?

19. The work done by the explosion of the mixture of 1 cu. ft. of coal gas with x cu. ft. of air is $W = 83x - 3.2x^2$. Find the value of x that makes W a maximum.

20. The speed of a point on a fly-wheel t seconds after starting was $s = 32t^2 - t^4$. Determine whether the speed was increasing or decreasing when $t = 2$, $t = 3$, $t = 4$, $t = 5$. For what value of t is the speed a maximum?

21. A piece of cardboard 12 inches square is to be made into an open-top box of maximum volume by cutting out equal squares at the corners and folding up the projecting sides. Find (a) the dimensions of the box. Find the dimensions and volume of the box if the cardboard is (b) 23″ square; (c) 8″ by 6″; (d) 30″ by 14″.

22. What are the most economical proportions for a cylindrical tin cup?

23. Find the dimensions of the maximum rectangle that can be inscribed in a triangle of base b and altitude h, two sides of the rectangle being perpendicular to the base of the triangle.

24. A sheet of paper is to contain 18 square inches of printed matter. The margins at the top and bottom are to be 2 inches each, and at the sides 1 inch each. Determine the dimensions of the page that will require the least amount of paper.

25. Find the coordinates of the maximum points on the curve $y = 18x^2 - x^4$. The coordinates of the minimum point.

26. The height (y ft.) of the cable of a suspension bridge above the water is $y = 18 + .04x^2$. Find the slope (a) at any point, (b) at the point where $x = 25$ ft.

27. The height of a ball t sec. after being thrown vertically upward was $h = 160t - 16t^2$. When was the ball highest? How high did it rise? When was it 384 feet high?

28. For a package to go by parcel post the sum of its length and girth must not exceed 7 feet. What are the dimensions and volume of the largest rectangular package with square ends that can go by parcel post?

SUGGESTION. $4s + l = 7$, $V = s^2l = s^2(7 - 4s)$.

148. Related Changes in Function and Variable. In problems dealing with areas or volumes, to each small

change or error in a linear measurement (the independent variable) there will be a corresponding change or error in the surface or in the volume, that is, in the function.

EXAMPLE. A side of a square was given as 10.04 inches instead of the correct value, 10 inches. Determine the resulting error in area. Area computed from measurement $= (10 + .04)^2 = 100 + .8 + .0016$

$$\text{True area} = 100.$$

$$\text{Resulting error} = .8 + .0016 = .8016$$

The value .8 is called the approximate error.

If the true length of the side of a square is x and the error in the reading is Δx, then $(x \pm \Delta x)^2 = x^2 \pm 2x\Delta x + (\Delta x)^2$. The error is then $2x\Delta x + (\Delta x)^2$. Now if Δx is small in comparison with x, the term $(\Delta x)^2$ can be dropped, giving a first approximation. The error in the area of the square is then said to be $2x\Delta x$, approximately. Hence if $A = x^2$, then the error in A is taken as $\Delta A = 2x\Delta x$, approximately. In like manner, if $V = x^3$ (volume of a cube), then the approximate change in volume corresponding to a change Δx is $\Delta V = 3x^2\Delta x$. Similarly, if $V = 4\pi r^3/3$, then the change in volume corresponding to a change in the radius Δr is $\Delta V = 4\pi r^2\Delta r$, approximately. These and similar formulas for the approximate error are most easily obtained as follows. From the definition of derivative,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

When Δx is small,

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} \quad \text{(approximately)}, \tag{20}$$

or

$$\Delta y = \frac{dy}{dx}\Delta x \quad \text{(approximately)}. \tag{21}$$

Applying this formula to the examples $A = x^2$, $V = x^3$, and $V = 4\pi r^3/3$ above, we get the formulas for the approximate errors given above.

149. Instantaneous Rate of Change. In problems dealing with falling bodies, speed of trains, energy, etc., the time-rate of change of one quantity can be expressed in terms of the time-rate of change of a related quantity. Consider the relation of the circumference of a circle to its radius, and suppose c takes on a small increment Δc when the corresponding increment in r is Δr. Then we have

$$c = 2\pi r, \quad \text{and} \quad c + \Delta c = 2\pi(r + \Delta r).$$

Subtracting, we find $\Delta c = 2\pi \Delta r$. If Δt is the time required for this change, then the average rate of increase in c in the time Δt is

$$\frac{\Delta c}{\Delta t} = 2\pi \frac{\Delta r}{\Delta t}.$$

That is, the average time-rate of change in the circumference $\Delta c/\Delta t$ is 2π times the average time-rate of change in the radius $\Delta r/\Delta t$. If the time Δt be made to approach zero as a limit, we find that the instantaneous time-rate of change of the circumference is

$$\frac{dc}{dt} = 2\pi \frac{dr}{dt}.$$

General Statement. *At any instant the time-rate of change of a function as $f(x)$, where x varies with the time, equals the derivative of the function multiplied by the time-rate of change of x.*

Example. The radius of a circular water wave is 8 ft. and is increasing at the rate of 4 ft. per second. How fast is the area increasing?

Solution.

$$A = \pi r^2.$$

The rate of increase in the area is dA/dt, and dr/dt is the rate of increase in the radius. Since $dA/dr = 2\pi r$, we have

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Since $r = 8$, and $dr/dt = 4$, we find

$$\frac{dA}{dt} = 2\pi 8 \times 4 = 64\pi.$$

Hence the area of the circle is increasing at the rate of 64π square feet per second.

150. Falling Bodies. If we wish to know the velocity or acceleration of a falling body at any instant, we can proceed as in § 141. The distance s feet passed over in t seconds by a body falling from rest is given by the equation

$$s = 16t^2. \tag{22}$$

The distance passed over in $t + \Delta t$ seconds is

$$s + \Delta s = 16(t + \Delta t)^2. \tag{23}$$

Subtracting (1) from (2),

$$\Delta s = 32t\Delta t + 16(\Delta t)^2,$$

in which Δs is the distance passed over in Δt seconds. Therefore

$$\frac{\Delta s}{\Delta t} = 32t + 16\Delta t,$$

which is the average velocity (speed) during Δt seconds. The limit approached as Δt approaches zero is $ds/dt = 32t$, the instantaneous velocity at any time t.

Hence, the velocity at the end of one second is 32 ft. per second; at the end of two seconds, 64 ft. per second, etc. Briefly, the first derivative of distance with respect to time is velocity; the first derivative of velocity with respect

to time is acceleration. Hence if the distance s is given by $s = 16t^2$, we may write

$$\text{velocity} = \frac{ds}{dt} = 32t, \quad \text{or} \quad v = 32t, \text{ in feet per second,}$$

$$\text{acceleration} = \frac{dv}{dt} = 32, \text{ in feet per second, per second.}$$

Similarly, if y equals the volume and x the edge of a cube which is increasing, we have $y = x^3$. Then the time-rate of change of y is

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt},$$

that is, the volume is increasing $3x^2$ times as fast as an edge.

Example. A body is thrown vertically upward with a velocity of 96 ft. per second. Find the velocity at the end of t seconds, (*b*) at the end of 2 seconds, (*c*) the time going up, (*d*) the highest point reached.

Solution. The equation is $s = 96t - 16t^2$. Then the velocity at the end of t seconds is

$$\frac{ds}{dt} = 96 - 32t,$$

and when $t = 2$, this is $96 - 64 = 32$. At the highest point the velocity is zero; hence $96 - 32t = 0$. Solving for t, $t = 3$; hence the time going up is 3 seconds. Setting $t = 3$, we find that the total distance upward is $s = 96 \times 3 - 16 \times 9 = 144$ ft.

EXERCISES.

1. If a metal cube expands so that its edge x increases from 10 inches to 10.1 inches, approximately what is the change in the volume?

2. Find the approximate change in the volume of a ball-bearing if it wears down from a sphere whose radius is 5 mm. to one whose radius is 4.97 mm. $V = 4\pi r^3/3$.

3. Find the instantaneous rate of change of the area of a square when the side b inches in length is increasing at the rate of a inches per second.

Suggestion. $A = x^2$, $dA/dt = 2x\, dx/dt$, also $x = b$, $dx/dt = a$.

4. Assuming that the volume of wood in a tree is proportional to the cube of its diameter and that the latter increases uniformly year by year when growing, show that the rate of increase in volume when the diameter is 2 ft. is 36 times as great as when the diameter is 4 inches.

SUGGESTION. $V = kd^3$.

5. The height (y ft.) of a projectile thrown vertically upward is given by the equation $y = 112t - 16t^2$, where t is the time in seconds. Find the velocity at any time. Find the velocity when $t = 2$, $t = 3$, $t = 5$. What does a negative velocity mean?

6. A stone dropped into a placid pond causes a series of concentric circular ripples. If the radius of the outer circle increases steadily at the rate of 6 ft. a second, how rapidly is the disturbed area increasing at the end of 2 seconds? 4 seconds?

7. If $y = kx^4$, find the rate of increase of y per unit change in x when $x = 3$.

8. Find approximately the error in the volume of a cube if the edge is measured 8.05″, instead of 8″.

9. A bomb is dropped from an airplane. Its height in feet above the ground t seconds later is given by $s = 3600 - 16t^2$. Find the rate of fall when $t = 2$ seconds. When $t = 5$ seconds. What will be its velocity when it reaches the ground?

10. A baseball is thrown directly upward with a velocity of 80 ft. per second. Find its velocity at the end of t seconds. How high will it rise? The equation is $s = 80t - 16t^2$.

11. A ball is rolled down a certain inclined plane. Its distance in feet from the starting point is given by $s = t^2/4 + 2t$. Find (a) its velocity at any time, (b) its velocity when $t = 3$ seconds.

12. A cliff 240 ft. high overhangs a river. A man on the edge of the cliff throws a stone vertically upward with a velocity of 32 ft. per second. How high will it rise? After how many seconds will it strike the water?

13. The kinetic energy of a body of mass m moving with a velocity v is given by $K = mv^2/2$. Find the rate at which the kinetic energy changes with respect to the change in velocity.

14. Find the derivatives of the following quantities. Substitute some numerical value and interpret each as a statement about rates.

(a) The distance a body falls in t seconds is $s = gt^2/2$.

(b) The kinetic energy of a moving body is $E = 10v^2$.

(c) The centrifugal force (F tons) of a locomotive as it rounds a curve at a velocity of v miles per hour is $F = .09v^2$.

15. If the speed of a train t seconds after starting is given by $v = 6t - .3t^2$, find the acceleration at any time. Find the acceleration when $t = 1, 3, 5, 10$.

16. The force (F lb.) applied to an object varied thus with the time $F = 75t - t^3$. Find how fast F is increasing when $t = 1, 2, 3, 4$.

17. The velocity of a point on a fly-wheel t seconds after starting was $v = 24t^2 - t^3$. Was this increasing or decreasing when $t = 2, 5, 10, 16, 20$?

18. In t seconds after brakes were applied a train moved a distance (s ft.) given by $s = 88t - 2t^2$. Find how fast it was moving when $t = 2, 3, 5, 22$. Also how fast the speed was decreasing.

19. The distance traveled by an object in t seconds was $s = 50t^4 - t^5$. Find when the speed was a maximum.

20. The kinetic energy of a train t minutes after starting was $E = 200t^2 - 20t + 6$. Find when the kinetic energy was a minimum.

21. The volume of a cube was increasing at the rate of 432 cu. in. per minute at the instant when the edge was 12 inches. How fast was the edge changing?

22. A sphere is expanding at the rate of 36π cu. inches per minute. Find how fast the radius and surface are expanding when $r = 3$.

23. A bullet was fired vertically. After t seconds the height s feet was $s = 800t - 16t^2$. With what velocity did it start upward? What was its greatest height? After how many seconds did it strike the ground?

24. How would you proceed to find the rate at which the velocity is changing if given the formula for distance? To find the acceleration if the formula for velocity is given?

25. Find how fast the volume of a cube is increasing if the edge is 4 inches and is increasing at the rate of $\frac{1}{2}$ inch per second.

26. A spherical rain-drop is gathering moisture in such a way that its radius is increasing at the rate of 1 mm. per minute. How fast is the volume of the rain-drop increasing when its radius is 1 millimeter? 2 millimeters? 4 millimeters?

27. A soap bubble of spherical form is increasing in volume at the uniform rate of 24 cubic inches per second. How fast is the radius increasing at the instant the radius is 3 inches? 4 inches?

151. Derivative of a Sum. *The derivative of the algebraic sum of a finite number of functions is equal to the like algebraic sum of the derivatives.* This has been assumed in the preceding work, but a formal proof will now be given.

PROOF. (Two functions.) Let

$$y = u + v, \tag{24}$$

where u and v, and hence y, are functions of the independent variable x. If x takes on an increment Δx, then u, v, and y will take on increments Δu, Δv, and Δy, respectively. Then we may write

$$y + \Delta y = u + \Delta u + v + \Delta v. \tag{25}$$

Subtracting (24) from (25), we have

$$\Delta y = \Delta u + \Delta v.$$

Dividing by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}. \tag{26}$$

Let Δx approach the limit zero; then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \tag{27}$$

Hence

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}. \tag{28}$$

A similar proof holds for the sum of any number of functions.

152. Derivative of a Product. *The derivative of the product of two functions is equal to the first times the derivative of the second, plus the second times the derivative of the first.*

PROOF. Let

$$y = uv. \tag{29}$$

Then, by the same reasoning as in § 151,

$$y + \Delta y = (u + \Delta u)(v + \Delta v) \tag{30}$$
$$= uv + u\Delta v + v\Delta u + \Delta u\Delta v.$$

Subtracting (29) from (30), we have

$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v.$$

Dividing by Δx, we get

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v. \tag{31}$$

If Δx approaches the limit 0, then Δu approaches 0, Δv approaches 0, and hence $\frac{\Delta u}{\Delta x}\Delta v$ approaches 0. Hence

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{32}$$

Therefore

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{33}$$

153. Derivative of a Quotient. *The derivative of a quotient (fraction) is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Consider

$$y = \frac{u}{v}. \tag{34}$$

Then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}, \tag{35}$$

and

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}. \tag{36}$$

Dividing by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}. \tag{37}$$

As Δx approaches 0, we find

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \tag{38}$$

Since any negative power may be written as a fraction, we may write

$$y = x^{-m} = \frac{1}{x^m}.$$

Using (38), with $u = 1$, $v = x^m$, we find

$$\frac{dy}{dx} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1}.$$

This shows that Rule III, § 144, holds also for negative integral powers. Such negative powers can therefore be differentiated either by Rule III or by expressing them as fractions and then using (38).

154. Summary. The above formulas are here collected and numbered for convenience of reference. See also § 144.

V. $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

VI. $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

VII. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

ILLUSTRATIONS.

EXAMPLE 1. Find the derivative of $y = x^2(x + 2)$ with respect to x.

SOLUTION. Since $y = x^2(x + 2)$ (a product),

$$\frac{dy}{dx} = x^2 + (x + 2)2x$$

$$= 3x^2 + 4x.$$

EXAMPLE 2. Find the derivative of $y = \frac{x^3}{1 + x^2}$.

SOLUTION. Since $y = \frac{x^3}{1 + x^2}$ (a quotient),

$$\frac{dy}{dx} = \frac{(1 + x^2)\,3x^2 - x^3\,2x}{(1 + x^2)^2}$$

$$= \frac{3x^2 + x^4}{(1 + x^2)^2}.$$

EXERCISES

1. Find the derivative of each of the following functions with respect to x.

(*a*) $y = x^3 + 2x^2 + 6$.

(*b*) $y = x^4 + 3x + 7$.

(*c*) $y = x^3 - 4x^2 + 5x$.

(*d*) $y = x^3(x - 2)$.

(*e*) $y = (x^3 + 2)(x + 1)$.

(*f*) $y = (x^2 + 5x)x^2$.

(*g*) $y = x^2 - x^{-2}$.

(*h*) $y = x^{-2}(1 + x + x^2)$.

(*i*) $y = \frac{x}{x + 3}$.

(*j*) $y = \frac{3}{x^2 + 4}$.

(*k*) $y = \frac{3}{x^3}$.

2. Find the slope of the line in each of the following exercises.

(*a*) $y = 2x + 3$.

(*b*) $3x - 5y = 10$.

(*c*) $2x + 5y = 12$.

(*d*) $y = x \tan 40° + 2$.

(*e*) $x \sin 20° - y \cos 20° = 4$.

(*f*) $3x - 6 = 7y$.

(*g*) $\frac{x}{3} - \frac{y}{2} = 1$.

(*h*) $y = 10$.

(*i*) $\frac{x}{4} + \frac{y}{5} = 1$.

3. Find the slope of the tangent to each of the following curves at the point given. Write the equation of the tangent at the given point.

(*a*) $y = x^2 + 6$ at (2, 10).

(*b*) $y = x^2 - 4x + 4$ at (2, 0).

(*c*) $y = 4x^2$ at (1, 4).

(*d*) $y^2 = 4x$ at (1, 2).

4. Find the slope of $y = x/(1 + x^2)$ at the origin.

5. Divide 50 into two parts such that the sum of the squares is a minimum.

6. Find the least value of the function $y = x^2 - 6x + 8$.

7. Find the greatest value of $y = 3x - x^3$.

8. Find the points where the curve $y = x^3 - 9x^2 + 24x$ is parallel to the x-axis.

9. A 400-gallon open-top tank is to be built with a square base and vertical sides, and is to be lined with copper. Find the proportions that will require the least copper. 1 gallon = 231 cu. in.

10. A cylindrical vessel with open top is to contain v cubic inches. Prove that the inner surface will be least if the height h is equal to the radius r.

SUGGESTION. $s = \pi r^2 + 2\pi rh$. $v = \pi r^2 h$.

11. Locate the vertex of each of the following parabolas by finding the point at which the slope is zero.

(a) $y = x^2 - 6x + 10$.
(b) $y = x^2 - 8x + 15$.
(c) $y = 6x - x^2$.
(d) $y = -25 + 10x - x^2$.
(e) $y = 10x - x^2$.
(f) $y = ax^2 + bx + c$.

12. A carpenter has 192 sq. ft. of lumber with which to build a box with square base and open top. Find the dimensions of the largest box he can make.

13. What is the ratio of the length of the legs of a right triangle when its area is a maximum, provided that the sum of the legs is constant?

14. Find the slope of the curve $y = 2x/(x^2 + 2)$ at the origin.

155. Interdependent Quantities. Related Rates. One quantity may depend on another directly, or through a third quantity that is related to both the others. Thus, if y depends on u, and u in turn depends on x, then a change Δx in x forces a change Δu in u, which in turn forces a change Δy in y. We may then write, if $\Delta u \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$

or, in the limit when Δx approaches zero,

VIIIa $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

EXAMPLE 1. If $y = (1 + x^2)^3$, find dy/dx. We may write

$$y = u^3, \qquad u = 1 + x^2.$$

Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 2x = 6x(1 + x^2)^2.$$

This may be checked by first expanding $(1 + x^2)^3$ and differentiating as in § 142.

EXAMPLE 2. If a sphere is expanding so that the radius r is increasing at the rate 0.1 in. per sec., find the rate at which the volume V is increasing when the radius is 20 in.

We have $V = 4\pi r^3/3$. Then, by VIIIa, we have

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \times 0.1 = 2\pi r^2/5,$$

since $dr/dt = 0.1$; hence, when $r = 20$, the rate of increase of V, that is, dV/dt, is $(2\pi/5) \times 400 = 160\pi$, in cu. in. per sec.; that is, a little over 500 cu. in. per sec.

Another use of VIIIa occurs when two quantities are connected by an equation in which both quantities are raised to powers.

EXAMPLE 3. Given $x^2 + y^2 = 1$, find dy/dx.

We may differentiate each term of this equation with respect to one variable, say x, if we use VIIIa and write

$$\frac{d(y^2)}{dx} = \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Then the result of differentiating the given equation is

$$2x + 2y\frac{dy}{dx} = 0,$$

from which, solving for dy/dx, we find

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{-x}{\sqrt{1 - x^2}},$$

since $y = \sqrt{1 - x^2}$.

A formula very similar to VIIIa holds true if each of two variables x and y is expressed in terms of a third variable t. Then we may write, if $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \div \frac{\Delta x}{\Delta t},$$

or, in the limit, as Δt, and therefore also both Δx and Δy, approach zero,

VIIIb $$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}.$$

This formula is frequently used when each of two variable quantities depends upon and is expressed in terms of the *time*.

EXAMPLE 4. If $y = t^3$, and $x = t^2$, find dy/dx. We have

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 3t^2 \div 2t = \frac{3}{2}t.$$

By means of VIIIb, it is possible to prove Rule III, § 144, for *fractional* values of n. Thus, if $y = x^{3/2}$, we have

$$y = t^3, \qquad x = t^2,$$

as in the preceding example, and we may write

$$\frac{dy}{dx} = \frac{3}{2}t = \frac{3}{2}\sqrt{x} = \frac{3}{2}x^{1/2}.$$

Since, for $y = x^{3/2}$, the value of n in Rule III is $n = 3/2$, it is clear that the result just found is the same as that given by Rule III for $n = 3/2$. In the same way, the student may show that Rule III gives the correct answer for any other example that contains a fractional power.

EXERCISES

1. Find the derivative dy/dx for each of the following cases.

(*a*) $y = (1 + x^2)^5$. (*b*) $y = (1 + 2x)^4$. (*c*) $y = (2 - 3x^2)^3$.

2. Find the derivative dy/dx for each of the following cases.

(*a*) $y = x^{5/2}$. (*c*) $y = x^{3/2} + 2x^{1/2}$. (*e*) $y = (1 + 2x)^{1/2}$.
(*b*) $y = x^{1/2}$. (*d*) $y = x^{2/3}$. (*f*) $y = (1 + x^2)^{1/2}$.

3. Find the derivative dy/dx for each of the following cases.

(*a*) $y^2 = x^3$. (*c*) $y = t^2, x = t^3$.
(*b*) $x^2 - y^2 = 1$. (*d*) $y = 1 + 2t, x = 3t + t^2$.

4. A circular metal plate expands by heat so that its radius increases uniformly at the rate of .02 inch per minute. At what rate is the surface increasing when the radius is 4 inches?

5. The edges of a cube are 12 inches and are increasing at the rate of .01 inch per minute. At what rate is the volume increasing?

6. In a parabola $y^2 = 4x$, if x increases uniformly at the rate of 6 inches per second, at what rate is y increasing when $x = 4$ inches?

7. The side of a square, 15 inches long, is increasing at the rate of 2 inches per second; at what rate is the area changing?

8. The abscissa of a point on the circle $x^2 + y^2 = 100$ is 6 and is increasing at the rate of 4 units per second. How fast is the ordinate (y) changing?

9. A ladder 25 feet long leans against a vertical wall. If the foot is pulled away at the rate of 5 feet per second, how fast is the top descending when 20 feet high?

10. An airplane flying horizontally at the rate of 60 mi. per hour passes over a fort at an elevation of 3 miles. How fast is its distance from the fort increasing 4 minutes later?

11. At a time when the elevation of the sun is 30°, a ball is thrown vertically to a height of 64 ft. How fast is the shadow of the ball moving along the ground just before it strikes the ground?

12. A baseball "diamond" is a square 90 ft. on each side. If a ball is batted along the third-base line with a speed of 60 feet per sec., how fast is its distance from first base changing 2 seconds later?

13. At what point on the parabola $y^2 = 8x$ are the abscissa and ordinate increasing at the same rate?

14. A man is walking at the rate of 5 mi. per hour toward the foot of a building 60 feet high. At what rate is he approaching the top of the building when he is 80 feet from its foot?

15. The volume (V cu. in.) of a quantity of gas varied according to the equation $V = 100/p$, where p is the pressure in lb./sq. in. If p increased at the rate of .1 lb. per sq. in. per second, how fast was V changing when $p = 20$?

$\frac{dD}{dt} = \frac{dD}{dx} \times \frac{dx}{dt}$ $\quad \frac{dD}{dt} = \frac{60 \cdot x}{\sqrt{8100 + x^2}}$

$D^2 = 90^2 + x^2$ $\quad \ldots$ when $x = 120$

$D = \sqrt{8100 + x^2} = (8100 + x^2)^{1/2}$ $\quad \frac{60 \times 120}{} = \frac{7200}{}$

(7) $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$ $2(15) \times 2 = 60$

$A = x^2$

$\frac{dy}{dx} = 2x$

CHAPTER X

INTEGRATION

156. The Differential Notation. The student is already familiar with the meaning of the symbols Δx, Δy and dy/dx. Certain new symbols are now to be defined in terms of these familiar symbols. If y is a function of x, then dy (read differential y) will be defined by the equation

$$dy = \frac{dy}{dx}\Delta x. \tag{1}$$

If we put $y = x$, then $dy/dx = 1$, and this definition reduces to the equation

$$dx = \Delta x. \tag{2}$$

Substituting dx (differential x) for its equal Δx in the definition, we get

$$dy = \frac{dy}{dx}dx. \tag{3}$$

Thus, if $y = x^2$, then $dy/dx = 2x$, and $dy = 2xdx$, and the reason for the use of the symbol dy/dx for the derivative is now evident.

Since $dy = (dy/dx)dx$, no new formulas are needed for finding differentials. When the derivative of the function has been found it has simply to be multiplied by the differential of the independent variable (usually dx) to give the differential of the function.

157. To Find a Function, its Derivative Being Given. The derivatives of several type functions have been considered, including u^n, $u + v$, uv, and u/v. Applications have been made in finding slopes, rates, maximum and minimum values, etc.

$\frac{dA}{dx} = 2x \cdot \frac{1}{2}(100 + x^2)^{-\frac{1}{2}}$

$= \frac{x}{\quad}$

The inverse process of finding a function when its derivative or differential is known leads to the solution of many important problems. This process is called integration. The function thus found is called the integral of the given derivative or differential. The sign of integration is $\int$. When this sign stands to the left of a differential, it indicates that a new function is to be found, whose differential is the expression to the right of the sign of integration. Thus, if

$$\frac{dy}{dx} = 3x^2,$$

then

$$y = x^3 + c,$$

which may also be written in the form

$$\int 3x^2 dx = x^3 + c.$$

Similarly,

$$\int 4x^3 dx = x^4 + c,$$

where c is a constant whose value is undetermined. When integrating, the arbitrary constant c is always used since the terms involving the variable (say x) determine only the amount of increase. The total value of the function at any time equals the increase plus the original value.

NOTE. The word *integrate* means *to make whole, to restore to its entirety, to give the sum or total.* The function to be integrated is called the integrand.

158. Geometric Significance of the Constant of Integration. The following example will illustrate what is meant.

ILLUSTRATIVE EXAMPLE. Find the equation of a curve whose slope at every point is equal to twice the abscissa of the point.

The problem is to find a function y such that

$$\frac{dy}{dx} = 2x.$$

It appears at once that $2x$ is the derivative of x^2. Integrating, we have

$$y = x^2 + c.$$

Since the derivative of a constant is zero, it is clear that whatever value c may have $dy/dx = 2x$. Hence the condition of the problem is satisfied by any one of the family of parabolas $y = x^2 + c$. If the curve, Fig. 98, passes through the point (2, 4), then $4 = 4 + c$ and $c = 0$. The equation is $y = x^2$. If the curve passes through the point (1, 7), then $7 = 1 + c$, and $c = 6$. The equation is $y = x^2 + 6$. If the curve passes through (0, -3), then $-3 = 0 + c$, and $c = -3$ The equation is $y = x^2 - 3$.

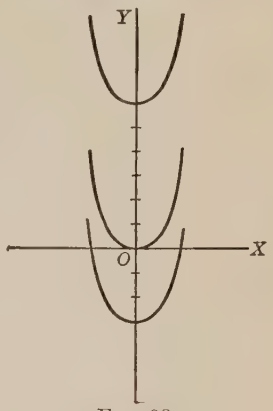


FIG. 98.

159. Integration Formula. Since the differentiation of any power of x leads to the next lower power, integration of any power of x leads to the next higher power. Thus, if

$$\frac{dy}{dx} = x, \quad \text{then} \quad y = \frac{x^2}{2} + c. \tag{4}$$

If

$$\frac{dy}{dx} = x^n, \quad \text{then} \quad y = \frac{x^{n+1}}{n+1} + c, \quad (n \neq -1). \tag{5}$$

By using (5) as a formula, various powers of x can be integrated at sight. If the denominator consists of x raised to a power, it should be brought into the numerator, the sign of the exponent being changed. Formula (5) is true for all real values of n except $n = -1$.

NOTE. Differentiation is a direct process. By means of the fundamental formulas the derivative of any elementary function may be found. The problem of finding the integral of a given expression is not so easily solved, and in the last analysis is a matter of trial. There are many comparatively simple functions whose integrals cannot be

expressed in terms of elementary functions. The student will find that the greatest aid in finding integrals is a thorough knowledge of the results of differentiation.

EXERCISES

1. Given the following values for dy/dx, find the value of y. Check the results by differentiation.

(a) $\frac{dy}{dx} = x,\ x^4,\ x^7,\ 3x^2,\ 5x^4,\ 4x^3,\ 6x^5,\ nx^{n-1}$.

(b) $\frac{dy}{dx} = -7x,\ x^{1/2},\ -5x^{3/2},\ -6x^2,\ -3x^2,\ -8x^3$.

(c) $\frac{dy}{dx} = x^{-6},\ x^{-10},\ x^{-a},\ x^{-1/2},\ x^{1/3},\ x^{1/4},\ x^{1/n}$.

(d) $\frac{dy}{dx} = x^5,\ x^{-4},\ x^{1/3},\ x^a,\ x^2 + 1,\ x^2 + 2x + 6$.

2. Find the value of y in each of the following cases.

(a) $\frac{dy}{dx} = x^2 + x + 3$.

(b) $\frac{dy}{dx} = 4x^3 + 3x^2 + 2x + 1$.

(c) $\frac{dy}{dx} = 8x + 12$.

(d) $\frac{dy}{dx} = (x^2 + 2)2x$.

(e) $\frac{dy}{dx} = (x^3 - 1)3x^2$.

(f) $\frac{dy}{dx} = (x^3 + 2x^2)(3x^2 + 4x)$.

3. Find the values of each of the following expressions.

(a) $\int x^3 dx$.

(b) $\int (3x^2 + 5x + 6)dx$.

(c) $\int (x^6 + 5x^3 + 7)dx$.

(d) $\int (x^{-2} + x^{-3})dx$.

(e) $\int x^{1/2} dx$.

(f) $\int x^{1/4} dx$.

4. Find the values of each of the following expressions.

(a) $\int 6x^5 dx$.

(b) $\int 10x^9 dx$.

(c) $\int x dx$.

(d) $\int x^{1/2} dx$.

(e) $\int x^{-3} dx$.

(f) $\int dx$.

(g) $\int \frac{1}{x^6} dx$.

(h) $\int (x^2 + 3x)dx$.

(i) $\int (1 + x)^2 dx$.

(j) $\int (\sqrt{x} + \sqrt[3]{x})dx$.

(k) $\int (2x + 3)dx$.

(l) $\int (x^2 + 5x + 7)dx$.

5. Verify each of the following statements.

(a) $\int ax^5 dx = a\frac{x^6}{6} + c$.

(b) $\int (x^2 + a)dx = \frac{x^3}{3} + ax + c$.

(c) $\int (x + 5)dx = \frac{x^2}{2} + 5x + c.$

(d) $\int (x^2 + 2)^2 dx = \frac{x^5}{5} + \frac{4x^3}{3} + 4x + c.$

(e) $\int (x^2 + x + c)dx = \frac{x^3}{3} + \frac{x^2}{2} + cx + c_1.$

(f) $\int (mx + 3)dx = \frac{mx^2}{2} + 3x + c.$

(g) $\int (x^2 + 5)dx = \frac{x^3}{3} + 5x + c.$

(h) $\int (3x^2 + 2x)dx = x^3 + x^2 + c.$

(i) $\int \left(x^2 + \frac{1}{x^2}\right) dx = \frac{x^3}{3} - \frac{1}{x} + c.$

(j) $\int (4x^3 + 6)dx = x^4 + 6x + c.$

6. The slope of a curve is given by $dy/dx = 2x + 3$. Find the equation of the curve if it passes through the point $x = 1$, $y = 4$.

7. The slope of a curve is given by $dy/dx = 2x + 7$. Find the equation of the curve if it passes through the point (1, 8).

8. Find the equation of the curve whose slope is $2x - 4$ and which passes through the point (1, 5). Plot the curve.

9. Find the equation of the curve which passes through the origin, if the slope is $4x - 8$.

10. A curve which passes through the point (2, 4) has for its slope $2x - 6$. Find the equation of the curve, and the coordinates of the minimum point. Plot the graph.

11. A body moves so that its velocity at any time (t) is given by $v = ds/dt = 16 - 2t$. Express the distance s as a function of t, given that $s = 0$ when $t = 0$. Compute the distance when $t = 6$.

12. Find the equation of the family of curves whose slope at every point is equal to the abscissa of that point. Equal to the abscissa + 2. Equal to the abscissa − 4. Equal to the square of the abscissa.

13. A particle slides down a smooth inclined plane so that $v = ds/dt = 10t$. How far does it slide in the time between $t = \frac{1}{2}$ and $t = 3$?

14. Find the equation of the system of curves such that the slope of the tangent at any point is:

(a) m. *Ans.* Straight lines, $y = mx + c$.

(b) x. Parabolas, $y = \frac{x^2}{2} + c$.

(*c*) $x - 3$. Parabolas, $y = \frac{x^2}{2} - 3x + c$.

(*d*) $3x^2$. Cubical parabolas, $y = x^3 + c$.

15. Assuming that $s = 5$ when $t = 0$, find the relation between s and t if the velocity ds/dt is: (*a*) 5, (*b*) $t + 3$, (*c*) $2t - 3$, (*d*) $3t^2 + 5$.

16. The slope of a certain suspension cable at any horizontal distance x from the origin (lowest point) is $.02x$. Find the equation of the curve made by the cable, and the height (y ft.) of any point, given that $y = 8$ when $x = 0$. Plot the curve from $x = 1$ to $x = 60$.

160. Distance, Velocity, Acceleration. The velocity v of an object moving uniformly is given by the equation

$$v = \frac{\text{distance}}{\text{time}} = \frac{s}{t}. \tag{6}$$

If the distance covered in equal intervals of time is variable, then s/t, which is merely the average velocity, may differ greatly from the velocity at a given time t_1. As we have seen, the velocity at any given instant is expressed by the equation

$$v = \frac{ds}{dt}, \tag{7}$$

that is, the time-rate of change in distance is velocity.

In like manner the acceleration a of an object whose velocity is changing uniformly is given by the equation

$$a = \frac{\text{change in velocity}}{\text{change in time}} = \frac{\Delta v}{\Delta t}.$$

If the acceleration is variable, then $\Delta v/\Delta t$, which is merely an average acceleration, may differ greatly from the acceleration at any given time t. The acceleration at any instant is given by the equation

$$a = \frac{dv}{dt}, \tag{8}$$

that is, the time-rate of change in velocity is acceleration. Formulas (7) and (8) are basic and should be memorized.

161. Falling Bodies. (See also § 150.) It is known from experiments that if an object is thrown vertically upward, in cases where the resistance of the air may be ignored, its velocity decreases at the rate of about 32 ft. per second in each second. This important constant is usually denoted by g. If the initial velocity is 64 ft. per second, the velocity after 1 second will be 32 ft. per second; after 2 seconds, zero ft. per second; after 4 seconds, $-$ 64 ft. per second, etc. Taking the direction upwards as positive, the acceleration or rate of change in velocity is $-$ 32 ft. per second per second, whether the object is rising or falling. Hence for bodies falling freely from rest, we have

$$\text{acceleration} = \frac{dv}{dt} = -g, \tag{9}$$

$$\text{velocity} = v = \frac{ds}{dt} = -gt, \tag{10}$$

$$\text{distance} = s = -\frac{1}{2}gt^2. \tag{11}$$

Example 1. If a heavy body is thrown downward with a velocity of 10 feet per second, find (a) its velocity, and the distance (b) passed over in 1 second; in 2 seconds; in 3 seconds.

Solution. Acceleration $= dv/dt = -32$.

Integrating, we have $v = -32t + c$. Since the body had an initial velocity of $-$ 10 feet per second, $v = -10$ when $t = 0$. Hence $-10 = 0 + c$, that is, $c = -10$. The equation is $v = -32t - 10$. The velocity per second at the end of 1 second is $-$ 42 ft.; 2 seconds, $-$ 74 ft.; 5 seconds, $-$ 170 ft. Now the velocity, which is ds/dt, equals $-32t - 10$. Integrating, we have

$$s = -16t^2 - 10t + c_1.$$

But $s = 0$ when $t = 0$; hence $c_1 = 0$, and the equation is

$$s = -16t^2 - 10t.$$

The distance passed over in 1 second is 26 ft.; in 2 seconds, 84 ft.; in 5 seconds, 450 ft.

EXAMPLE 2. From the top of a cliff 300 ft. high a body is projected vertically upward with a velocity of 20 feet per second. Find its velocity and its height 2 seconds later. Five seconds later.

SOLUTION. Acceleration $= dv/dt = -32$. Integrating, we find $v = -32t + c$. But $v = 20$ when $t = 0$. Hence $20 = 0 + c$, or $c = 20$, and

$$v = -32t + 20. \tag{1}$$

To find the distance, let $v = ds/dt$. Then

$$\frac{ds}{dt} = -32t + 20.$$

Integrating, we have

$$s = -16t^2 + 20t + c_1.$$

But $s = 300$ when $t = 0$; hence $c_1 = 300$ ft. Then the height at any time is

$$s = -16t^2 + 20t + 300. \tag{2}$$

From equations (1) and (2), we obtain:

If $t = 2$, $v = -44$ ft. per sec., and $s = 276$ ft.

If $t = 5$, $v = -140$ ft. per sec., and $s = 0$.

The height at any time is equal to the original height 300 ft. plus the distance passed over due to the initial velocity, minus the distance passed over due to gravity.

EXERCISES

1. Define (*a*) velocity, (*b*) acceleration.

2. A bomb is dropped from an airplane 8000 feet high. After how many seconds will it reach the ground?

3. An object is dropped from an altitude of 2 miles. Through what distance will it fall in 10 seconds? During the 12th second?

4. A body is projected upward with a velocity of 128 feet per second. How long will it be in the air? How high will it go?

5. A bomb, dropped from an airplane, struck the ground in 20 seconds. How high was the airplane?

6. On the moon the acceleration due to the moon's gravity is 5.3 feet per second^2. If on the moon a ball is projected vertically upward with a velocity of 100 feet per second, how high will it rise?

7. On the sun the acceleration due to gravity is approximately $27.6g$, or 883.2, feet per second^2. How far would a body fall in 3 seconds? In 5 seconds?

8. If a ball is thrown vertically upward with a velocity of 96 feet per second, find its velocity and height after t seconds. After 5 seconds. After 6 seconds.

9. From a point on a building 144 feet high, a body is projected upward with a velocity of 128 feet per second. Find its height at the end of 1, 2, 3, and 5 seconds. When will it reach the ground?

10. A body is projected downward with a velocity of 10 feet per second. Find its velocity at any time, and the distance covered in 4 seconds. In 5 seconds. In 10 seconds.

11. A body, starting from rest, moves according to the following equation, $dv/dt = 6$. Find (a) velocity, (b) distance at any time.

12. A body falls with a velocity $v = -32t + 12$. Find s in terms of t and find the distance passed over between the times $t = 1$ and $t = 6$.

13. A body is projected vertically with a velocity of v_0 feet per second from a point s_0 feet high. Find the equations for v and s and discuss the meaning of the constants.

14. The velocity of a certain moving body is given by the equation $v = 12 + 2t$. Find the distance passed over during the interval from $t = 0$ to $t = 5$.

15. If $v = 5t^2$, find the distance traveled during the 5th second.

16. If $v = t^3 + 2t^2 + t + 4$, find the distance traveled between the ends of the 2d and 5th seconds.

17. A torpedo, shot under water, has a speed of 40 feet per second at the instant its compressed air is exhausted. If it suffers a constant retardation of 5 feet per second per second, how far will it go before coming to rest?

SUGGESTION. $ds/dt = 40 - 5t$, $t = 8$ seconds.

18. A rifle bullet enters an embankment 21 feet thick with a velocity of 1000 feet per second. If it suffers a constant retardation of 50,000 feet per second per second, how far will it penetrate the embankment before coming to rest?

19. From an airplane 2400 feet high, a projectile is fired vertically upward with an initial velocity of 1200 feet per second. After how many seconds will it pass a balloon 8000 feet high? Find (a) when it will be highest, (b) when it will strike the ground. (Ignore air resistance.)

20. A point on the rim of a flywheel was moving at the speed of 48 feet per second when the power was cut off. The speed thereafter decreased at the rate of 6 feet per second^2. Find when the wheel stopped and how far the point moved before coming to rest.

162. Standard Elementary Forms. For the purpose of reference some standard elementary forms are given below.

I. $\int (du + dv + dw) = u + v + w + c.$

II. $\int a dv = av + c.$

III. $\int dx = x + c.$

IV. $\int v^n dv = \frac{v^{n+1}}{n+1} + c.$

The chief difficulty in integration is to recognize the type form of the integrand, and the factor to be used as dv. The student should observe carefully what part of the integrand is represented by v in the formula, and see that the corresponding dv is present. In selecting the factor dv it will sometimes be necessary to take out or introduce a constant factor under the integration sign. The selection of this factor is a matter of judgment and experience. The ability to recognize type forms is very valuable and should be carefully cultivated.

Thus we may write

$$\int x^4 dx = \frac{1}{5} x^5 + c;$$

$$\int \frac{dx}{x^3} = \int x^{-3} dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c;$$

$$\int (x^2 + 5) x dx = \frac{1}{2} \int (x^2 + 5) 2x dx = \frac{(x^2 + 5)^2}{4} + c.$$

To verify the result, find the derivative of the answer; this should be equal to the quantity under the integral sign.

NOTE. A variable must not be introduced under, or taken from under, the integral sign.

$$\int (ax^2 + b) x dx = \frac{1}{2a} \int (ax^2 + b) 2ax dx = \frac{1}{2a} \frac{(ax^2 + b)^2}{2} + c$$

$$= \frac{1}{4a} (ax^2 + b)^2 + c.$$

Here dv is $d(ax^2 + b)$ or $2axdx$.

EXERCISES

1. Find the values of each of the following integrals.

(a) $\int (x^2 + 2)2x dx$.
(b) $\int (x^3 + 5)3x^2 dx$.
(c) $\int (x^2 + 2x + 3)(2x + 2)dx$.
(d) $\int (x^3 + x^2)(3x^2 + 2x)dx$.
(e) $\int (x^2 + 4)2x dx$.
(f) $\int (x^3 + 7)3x^2 dx$.
(g) $\int \sqrt{x} dx$.
(h) $\int (6 + x)^2 dx$.

2. Verify each of the following integrations.

(a) $\int x^6 dx = \frac{x^7}{7} + c$.
(b) $\int ax^4 dx = a\frac{x^5}{5} + c$.
(c) $\int ax^{2/3} dx = \frac{3}{5} ax^{5/3} + c$.
(d) $\int 5y dy = \frac{5}{2} y^2 + c$.
(e) $\int 8t^{1/2} dt = \frac{16}{3} t^{3/2} + c$.
(f) $\int s^{-1/2} ds = 2s^{1/2} + c$.
(g) $\int \sqrt{2px} dx = \frac{2}{3} x\sqrt{2px} + c$.
(h) $\int 3ax^2 dx = ax^3 + c$.

3. Find the values of each of the following integrals.

(a) $\int (x^2 + 2)x dx$.
(b) $\int (x^3 + 3)x^2 dx$.
(c) $\int (x^2 + 4)^2 x dx$.
(d) $\int (x^3 + 5)x^2 dx$.
(e) $\int (x^2 + 2x)(x + 1)dx$.
(f) $\int (x^3 + 3x)^2 (x^2 + 1)dx$.
(g) $\int (x^4 + 8x)^3 (x^3 + 2)dx$.
(h) $\int (x^3 + 3x^2 + 6x)(x^2 + 2x + 2)dx$.

163. The Definite Integral. The integral of an expression has in general distinct values for given values of the variable. Finding the difference between two values of the integral corresponding to two values of the variable is called evaluating a definite integral.

Thus, if

$$\int x^2 dx = \frac{x^3}{3} + c$$

be evaluated for $x = 1$ and $x = 4$, and the first value subtracted from the second, we obtain

$$\frac{4^3}{3} + c - \left(\frac{1^3}{3} + c\right) = 21\frac{1}{3} - \frac{1}{3} = 21.$$

The values $x = 1$, and $x = 4$, are called the limits of integration, 4 being the upper limit and 1 the lower limit. It is customary to write

the above example as follows:

$$\int_1^4 x^2dx = \frac{x^3}{3} + c \Big]_1^4 = 21.$$

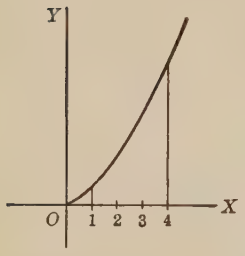


Fig. 99.

Example. Find the value of

$$\int_0^5 (2x^2 + 6)4xdx.$$

Solution.

$$\int_0^5 (2x^2 + 6)4xdx = \frac{(2x^2 + 6)^2}{2} + c \Big]_0^5$$

$$= \frac{3136}{2} + c - \left(\frac{36}{2} + c\right).$$

$$= 1550.$$

Note. In the process of subtraction in each of the above examples the constant c was eliminated. This will always occur and it is therefore unnecessary to write the constant in evaluating a definite integral. An integral in which the undetermined constant of integration c appears is called an indefinite integral. The integral in the above exercise is a definite integral.

164. The Area under a Curve. Consider the area A ($RSTH$, Fig. 100) bounded by the curve $y = f(x)$, the

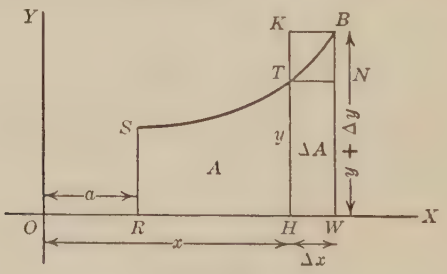


Fig. 100.

x-axis, the fixed ordinate RS at $x = a$, and the moving ordinate HT. Since the area A changes as x changes and is fixed when x is fixed, A is a function of x. As H moves to W, x increases by an amount Δx, A by an amount ΔA ($HTBW$) and y by an amount $NB = \Delta y$.

Since $HTBW$ is less than the rectangle $HKBW$ and greater than the rectangle $HTNW$, we have

$$HTNW < HTBW < HKBW \tag{12}$$

or

$$y\Delta x < \Delta A < (y + \Delta y)\Delta x. \tag{13}$$

Dividing by Δx, we have

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y. \tag{14}$$

As Δx approaches 0, $y + \Delta y$ approaches y and $\Delta A/\Delta x$ approaches dA/dx. Since $\Delta A/\Delta x$ lies *between* y and a quantity which approaches y, it also approaches y and we have

$$\frac{dA}{dx} = y, \quad \text{or} \quad dA = ydx. \tag{15}$$

Hence

$$A = \int ydx. \tag{16}$$

Before integrating, the value of y in terms of x must be substituted under the integral sign. The constant of integration can be evaluated by remembering that $A = 0$ when $x = a$, or a definite integral can be used.

The method of procedure will be illustrated by the solution of a typical example.

EXAMPLE. Find the area bounded by the curve $y = x^2 + 3$, the axis of x and the ordinates $x = 0$ and $x = 4$.

SOLUTION.

$$A = \int_0^4 ydx = \int_0^4 (x^2 + 3)dx = \frac{x^3}{3} + 3x\Big]_0^4 = \frac{64}{3} + 12 - 0 = 33\tfrac{1}{3}.$$

Similarly $\int_1^4 x^2dx$, evaluated in § 163, gives the area under the curve $y = x^2$ from $x = 1$ to $x = 4$. See Fig. 99.

EXERCISES

Evaluate the following:

1. $\int_0^2 x^2 dx$.
2. $\int_1^3 (2x^2 - 5)dx$.
3. $\int_1^5 (2x + 7)dx$.
4. $\int_0^4 (x^3 + 2)dx$.
5. $\int_0^3 (x^2 + x + 1)dx$.
6. $\int_1^5 x^{-2} dx$.
7. $\int_0^2 (x + 7)dx$.
8. $\int_0^1 5x dx$.
9. $\int_2^3 (x^4 - 3)dx$.
10. $\int_3^4 (x^2 - 5x + 1)dx$.
11. $\int_1^6 (x^3 - x^2 + x + 2)dx$.
12. $\int_4^7 (3x - 10)dx$.

13. Plot the line $y = 2x + 4$ from $x = 0$ to $x = 10$. From your knowledge of geometry find the area under it. Find this area by integration.

14. Find by integration the area bounded by the line $y = x + 6$, the x-axis, and the ordinates corresponding to $x = 1$ and $x = 5$. Can you check your result by geometry?

15. Find the area under $y = x + 5$ from $x = 0$ to $x = 4$.

16. Find the area under $y = x^2 + 1$ from $x = 0$ to $x = 5$.

17. Find the area under $y = x^2 + 3$ from $x = 1$ to $x = 6$.

18. Find the area under $y = x^3$ from $x = 0$ to $x = 3$.

19. Find the area between the parabola $y = 4 - x^2$ and the x-axis.

20. Find by integration the area of the trapezoid bounded by the line $y = 3x + 5$, the x-axis, and the ordinates corresponding to $x = 1$, and $x = 5$. Verify your result by finding the area as one-half the product of the sum of the parallel sides and the altitude.

21. Find the area bounded by the semi-cubical parabola $y^3 = x^2$, the x-axis, and the line $x = 8$.

22. Find the area bounded by the parabola $y^2 = 4x$, the x-axis, and the latus rectum $x = 1$.

23. Find the area bounded by the parabola $y^2 = 4x$, the x-axis, and the lines $x = 2$ and $x = 7$.

24. Find the area between the curve $y = 16 - x^4$ and the x-axis.

25. Find the area above the x-axis and below the parabola $y = -x^2 + 16$.

26. Plot the curve $y = 18 - 2x^2$ and find by integration the area under that part of the curve for which both x and y are positive.

27. Find the area in the first quadrant bounded by the curve $y^2 = 4x$ and the straight line $y = x$.

28. Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.

29. Compute the area bounded by the parabola $y^2 = 4x$, the x-axis, and the lines $x = 2$ and $x = 6$.

30. Show from known formulas of physics that the definite integral

$$\int_{t_1}^{t_2} gtdt$$

gives the distance traveled by a falling body in the time interval from $t = t_1$ to $t = t_2$.

CHAPTER XI

RELATIONS AMONG TRIGONOMETRIC FUNCTIONS

165. Introduction. The simplest formulas of trigonometry were given in § 34, and are restated here for convenience.

The *reciprocal relations* (§ 34) are

$$(1)\quad \operatorname{ctn} A = \frac{1}{\tan A}, \quad \sec A = \frac{1}{\cos A}, \quad \csc A = \frac{1}{\sin A}.$$

We found also (§ 34) the relations

$$(2)\qquad \tan A = \frac{\sin A}{\cos A}, \qquad \operatorname{ctn} A = \frac{\cos A}{\sin A};$$

and the *Pythagorean relations*

$$(3)\quad \sin^2 A + \cos^2 A = 1, \quad \text{or} \quad \cos^2 A = 1 - \sin^2 A;$$

$$(4)\quad 1 + \tan^2 A = \sec^2 A, \quad \text{or} \quad \tan^2 A = \sec^2 A - 1;$$

$$(5)\quad 1 + \operatorname{ctn}^2 A = \csc^2 A, \quad \text{or} \quad \operatorname{ctn}^2 A = \csc^2 A - 1.$$

All of these relations are true, as was remarked in § 70, for angles of any size. By simple transposition and other elementary operations of algebra, these relations may be written in many other forms. Thus, by clearing of fractions, the relations (1) may be written in the form

$$(6)\qquad \tan A \cdot \operatorname{ctn} A = 1, \qquad \cos A \cdot \sec A = 1,$$
$$\sin A \cdot \csc A = 1.$$

By taking the square root, we may write (3) in the forms

$$(7)\quad \cos A = \pm\sqrt{1 - \sin^2 A}, \quad \text{or} \quad \sin A = \pm\sqrt{1 - \cos^2 A}.$$

The sign plus holds in both cases if A is an acute angle,

since both $\cos A$ and $\sin A$ are then positive. For other angles, the correct sign can be determined by noting the proper sign for $\cos A$ or $\sin A$ in the quadrant in which the angle lies.

166. Algebraic and Trigonometric Transformations. Sometimes it is desirable to transform an expression involving trigonometric functions so as to express it in terms of a given function. It is to be noted that any identity which involves a fraction does not hold for values of the variables which make the denominator of the fraction zero. In an identity that contains a radical, the identity may be true only when the proper sign is attached to the radical, as was remarked in the case of formula (7) above.

EXAMPLE 1. Express $\sin^2 A - \cos^2 A$ in terms of $\sin A$.

Since $\cos A$ is to be expressed in terms of $\sin A$, the student should endeavor to recall a formula which will give $\cos^2 A$ in terms of $\sin A$. Obviously the formula is $\cos^2 A = 1 - \sin^2 A$. Substituting this value, we obtain

$$\sin^2 A - (1 - \sin^2 A).$$

Removing the parentheses, the expression reads

$$\sin^2 A - 1 + \sin^2 A.$$

Combining terms,

$$\sin^2 A - \cos^2 A = 2 \sin^2 A - 1.$$

EXAMPLE 2. Prove the identity $\cos x\sqrt{\sec^2 x - 1} = \sin x$.

SOLUTION. $\sqrt{\sec^2 x - 1}$ may be expressed in terms of $\tan x$, since

$$\tan^2 x = \sec^2 x - 1, \qquad \text{or} \qquad \tan x = \sqrt{\sec^2 x - 1}.$$

Substituting, we may write the given expression in the form

$$\cos x \tan x.$$

But we have

$$\tan x = \frac{\sin x}{\cos x}.$$

Making this substitution and reducing, we find

$$\cos x \frac{\sin x}{\cos x} = \sin x,$$

which was to be proved.

Starting now with the identity $\sin x = \sin x$, we may retrace each step, and thus reach the given identity. In proving an identity, it is necessary to see that the steps taken can be reversed, so as to return to the form that was to be proved.

EXERCISES

1. Find:

(*a*) $\sin x$ in terms of $\cos x$.
(*b*) $\cos x$ in terms of $\sin x$.
(*c*) $\tan x$ in terms of $\sec x$.
(*d*) $\operatorname{ctn} x$ in terms of $\csc x$.
(*e*) $\sec x$ in terms of $\tan x$.
(*f*) $\csc x$ in terms of $\operatorname{ctn} x$.

2. Prove each of the following identities.

(*a*) $\sin x \sec x = \tan x$.
(*b*) $\cos x \tan x = \sin x$.
(*c*) $\cos x \csc x = \operatorname{ctn} x$.
(*d*) $\sin x \operatorname{ctn} x = \cos x$.
(*e*) $\tan x + \operatorname{ctn} x = \sec x \csc x$.
(*f*) $(1 + \tan^2 x)\cos^2 x = 1$.
(*g*) $\sin^2 x = (1 - \cos x)(1 + \cos x)$.
(*h*) $\tan^2 x = (\sec x - 1)(\sec x + 1)$.
(*i*) $\tan^2 A - \sec^2 A = -1$.
(*j*) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$.

3. Prove each of the following identities.

(*a*) $\sin x = \cos x \tan x = \dfrac{\cos x}{\operatorname{ctn} x} = \dfrac{\tan x}{\sec x}$.

(*b*) $\cos x = \operatorname{ctn} x \sin x = \dfrac{\operatorname{ctn} x}{\csc x} = \dfrac{\sin x}{\tan x}$.

(*c*) $\tan x = \sin x \sec x = \dfrac{\sin x}{\cos x} = \dfrac{\sec x}{\csc x}$.

(*d*) $\operatorname{ctn} x = \csc x \cos x = \dfrac{\csc x}{\sec x} = \dfrac{\cos x}{\sin x}$.

(*e*) $\sec x = \tan x \csc x = \dfrac{\tan x}{\sin x} = \dfrac{\csc x}{\operatorname{ctn} x}$.

(*f*) $\csc x = \sec x \operatorname{ctn} x = \dfrac{\sec x}{\tan x} = \dfrac{\operatorname{ctn} x}{\cos x}$.

4. Prove each of the following identities.

(*a*) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$.
(*b*) $\sin^4 x \cos^2 x + \sin^2 x \cos^4 x = \sin^2 x \cos^2 x$.
(*c*) $\dfrac{1 + \sin x}{\cos x} = \dfrac{\cos x}{1 - \sin x}$.
(*d*) $\cos^3 x - \sin^3 x = (\cos x - \sin x)(1 + \sin x \cos x)$.
(*e*) $\sqrt{1 - \sin^2 x}\tan x = \sin x$.

5. Prove each of the following identities.

(*a*) $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$.

(*b*) $(\sec x + \tan x)(\sec x - \tan x) = 1$.

(*c*) $\sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \dfrac{\sin x}{1 + \cos x}$.

(*d*) $\sqrt{\dfrac{1 + \cos x}{1 - \cos x}} = \dfrac{1 + \cos x}{\sin x}$.

(*e*) $\cos^4 x - \sin^4 x = 1 - 2 \sin^2 x$.

(*f*) $\sin A \cdot \cos A(\tan A + \operatorname{ctn} A) = 1$.

(*g*) $(\sec^2 x - 1)(\csc^2 x - 1) = 1$.

(*h*) $\cos x \cdot \csc x \cdot \tan x = 1$.

6. Prove each of the following identities.

(*a*) $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$.

(*b*) $\sec^4 x(1 - \sin^4 x) = 2 \tan^2 x + 1$.

(*c*) $\sqrt{(1 + \tan^2 x)(1 + \operatorname{ctn}^2 x)} = \dfrac{1}{\sin x \cos x}$.

(*d*) $(1 + \operatorname{ctn}^2 A) \sin^2 A = 1$.

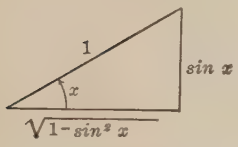


Fig. 101.

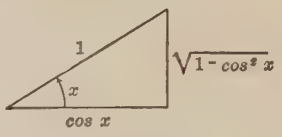


Fig. 102.

7. Express each of the trigonometric functions in terms of $\sin x$. See Fig. 101.

8. Express each of the trigonometric functions in terms of $\cos x$. See Fig. 102.

9. Express each of the trigonometric functions in terms of $\tan x$. See Fig. 103.

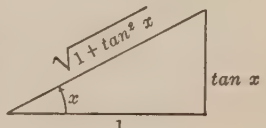


Fig. 103.

167. Graphical Representation. In the preceding chapters, we have represented graphically many algebraic relations, and we have found that they correspond to various curves, such as straight lines, circles, parabolas, etc.

Trigonometric functions may be represented by graphs. Such graphs often show the nature of the functions more clearly than do equations.

The simplest trigonometric function is sin x. The graph is readily plotted if the assumed value of the angle, expressed in radians, is taken as the abscissa of a point on the curve, and the corresponding value of the function as the ordinate.

168. The Sine Curve. Taking $y = \sin x$, let us assume values for x, as is done in the adjoining table, and compute the corresponding values of y. Plot these points. A smooth curve drawn through the points taken in order is called the graph of the function.

x in Degrees	x in Radians	y
0°	0	0
30°	$\pi/6$	0.50
45°	$\pi/4$	0.71
60°	$\pi/3$	0.87
90°	$\pi/2$	1.00
120°	$2\pi/3$	0.87
135°	$3\pi/4$	0.71
180°	π	0.00
225°	$5\pi/4$	−0.71
270°	$3\pi/2$	−1
315°	$7\pi/4$	−0.71
360°	2π	0

The graph is seen to be continuous and to consist of an infinite number of waves like those in Fig. 104. The curve is called the sine curve or sinusoid. The maximum value of sin x is seen to be 1, and the minimum value -1.

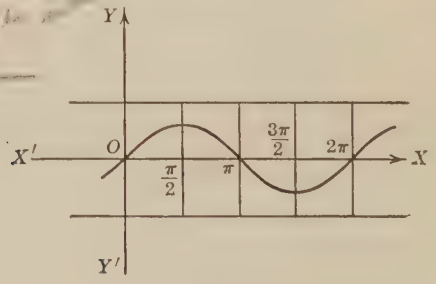


Fig. 104.

169. Zeros of sin x. Since the graph crosses the x axis at $0, \pm\pi, \pm 2\pi, \cdots, \pm n\pi, \cdots$, an infinite number of

times, the equation $\sin x = 0$ has an infinite number of roots, namely, $0, \pm \pi, \pm 2\pi, \cdots, \pm n\pi, \cdots$.

170. Period of sin x. Since for each value of x, $\sin(x + 2\pi) = \sin x$, the function $\sin x$ is said to be *periodic*, with 2π as a *period*.

171. The Cosine Curve. Taking $y = \cos x$, let us assume values for x, as was done in plotting $y = \sin x$, and obtain the corresponding values of y. Plot these points. The required graph, Fig. 105, is a smooth curve drawn through the points.

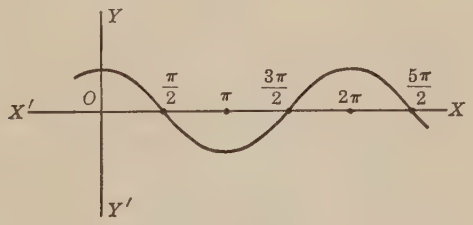


FIG. 105.

It will be seen that the graphs of $y = \sin x$, and $y = \cos x$ are identical in form, but the second is displaced to the left by $\pi/2$ units.

172. Wave Motion. If the motion of the point which generates the curve $y = \sin x$, or $y = \cos x$, be projected on the y axis, the projected motion is termed ***simple harmonic motion*** (S.H.M.), which is beautifully illustrated in the tuning fork.

In nature vibratory motion is exceedingly common. Two types are readily distinguishable. In one type each point of the vibrating body repeats its former motion. Such motion is found in the tuning fork, helical spring, and in violin strings. Rhythmic movements are also found in heart beats, respiration and the tides. In the second type the vibratory impulse is carried over an extent of space as well as time. Sound waves, light waves, heat waves, and electrical waves are of this character.

Both types of motion may be represented mathematically, the basic function being $y = \sin x$. Suppose the curve traced by a point on one prong of a tuning fork (Fig. 106) is $y = \sin 256\pi t$; then each complete

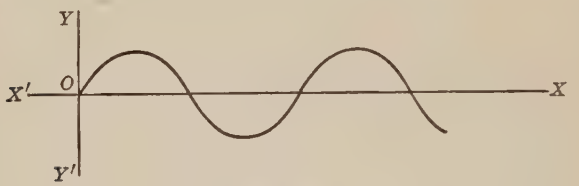


Fig. 106.

vibration would require 1/128 of one second. Since the generated sound wave travels 1100 feet per second at 44° Fahrenheit, the length of each wave is 1100/128 = 8.6 ft. If a helical spring is weighted, the vibratory motion will be similar to that of the tuning fork, but slower.

Light waves have a velocity of 300×10^6 meters per second (or 186,000 miles). Different wave-lengths correspond to different colors. The wave-length of the light from sodium (D_2 of the spectrum) is 589×10^{-9} meters. Hence n, the number of waves per second, is

$$n = \frac{300 \times 10^6}{589 \times 10^{-9}} = 509 \times 10^{12}.$$

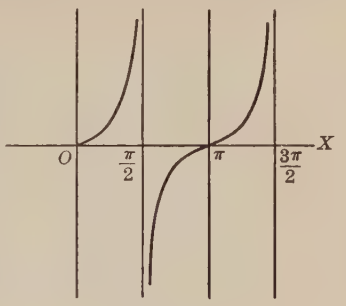
173. The Tangent Curve. The variation of tan x is readily shown by the graph of $y = \tan x$. Assume values for x and compute the corresponding values for y. Plot these points, using for x the corresponding values of the angles in radians. The graph through these points is seen

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	225°	270°
y	0	.58	1	1.73	∞	−1.73	−1	−.58	0	1	∞

to consist of an infinite number of branches. Each branch

lies between vertical lines π units apart. Thus the curve is discontinuous, the first discontinuity appearing where $x = \pi/2$ or 90°.

174. Reciprocal Relations. Using the method of the previous paragraph, the graphs of cotangent, secant, and cosecant are readily obtained. The graphs given below show the reciprocal relations between the pairs of trigonometric functions.

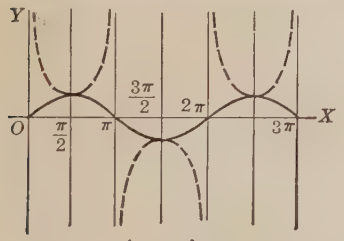


$y = \tan x$

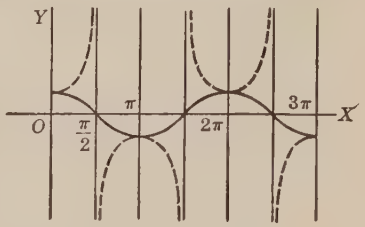
FIG. 107.

EXERCISES

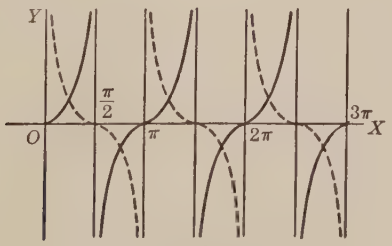
1. Does sin x increase or decrease when x increases from 0° to 90°? When x increases from 90° to 180°? When x increases from 180° to 270°?



sin x *and* csc x



cos x *and* sec x



tan x *and* ctn x

FIG. 108.

2. Does $\cos x$ increase or decrease when x increases from 0° to 90°? When x increases from 90° to 180°? When x increases from 180° to 270°?

3. By inspection find the coordinates of two or more maximum points on the curve $y = \sin x$. Of one or more minimum points. What is the greatest value of $\sin x$? What is the least?

4. By inspection find the coordinates of two or more maximum points on the curve $y = \cos x$. One or more minimum points. What is the greatest value of $\cos x$? The least value of $\cos x$?

5. Does a tangent curve have either a maximum or minimum point? Does $\tan x$ increase or decrease as x increases?

6. Draw the graphs of

$$(a)\ y = \sin 2x. \qquad (b)\ y = \cos 2x. \qquad (c)\ y = \sin 3x.$$

In what way does each of these graphs differ from the graph of the same trigonometric function of x?

7. Draw the graphs of

$$(a)\ y = \sin \frac{x}{2}. \qquad (b)\ y = \cos \frac{x}{2}. \qquad (c)\ y = \sin \frac{x}{3}.$$

8. Draw the graphs of

$$(a)\ y = 2 \sin x. \qquad (b)\ y = \sin^2 x. \qquad (c)\ y = 1 + \sin x.$$

9. Draw the graphs of

$$(a)\ y = \tan 2x. \qquad (b)\ y = \tan \frac{x}{2}. \qquad (c)\ y = \tan^2 x.$$

10. Draw the graphs of

$$(a)\ y = \sin (x + \pi/4). \qquad (b)\ y = \cos (x - \pi/4).$$

175. Functions of Sums and Differences. Thus far the relations between the functions of a single angle have been discussed. We next consider the functions of the sum or difference of two angles in terms of the functions of the angles themselves.

176. Sine and Cosine of the Sum of Two Angles. Let the angles x and y each be positive acute angles so chosen that $x + y$ is less than 90°. Lay off angle $XOM = x$, and

angle $MON = y$; then angle $XON = x + y$. From any point P in the terminal line ON, draw PD and PB perpendicular to OX and OM, respectively. From B draw BC perpendicular to PD, and BA perpendicular to OX. The triangles OAB and PCB are similar; hence $\angle CPB = x$.

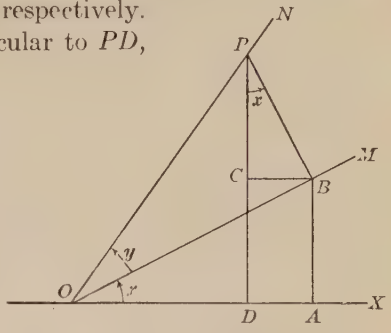


Fig. 109.

$$\sin (x + y) = \frac{DP}{OP} = \frac{DC + CP}{OP} = \frac{AB}{OP} + \frac{CP}{OP}.$$

Multiply both numerator and denominator of AB/OP by OB, the side common to the two triangles of which AB and OP are sides, respectively. Also multiply CP/OP in the same way by PB, the side common to the triangles PCB and OBP. Then

$$\sin (x + y) = \frac{AB}{OB} \times \frac{OB}{OP} + \frac{CP}{PB} \times \frac{PB}{OP},$$

or

(8) $$\sin (x + y) = \sin x \cos y + \cos x \sin y.$$

Likewise

$$\cos (x + y) = \frac{OD}{OP} = \frac{OA - DA}{OP} = \frac{OA}{OP} - \frac{CB}{OP}$$
$$= \frac{OA}{OB} \times \frac{OB}{OP} - \frac{CB}{PB} \times \frac{PB}{OP}$$

or

(9) $$\cos (x + y) = \cos x \cos y - \sin x \sin y.$$

In these proofs, if the sum $x + y$ of the acute angles x and y is obtuse, as in Fig. 110, the same arguments hold true, except that the sign of OD will be negative. If

$x + y = 90°$, or if $x = 0$ or $y = 0$, it is easy to show that the formulas hold by direct check. The above formulas are therefore true for all acute values of x and y.

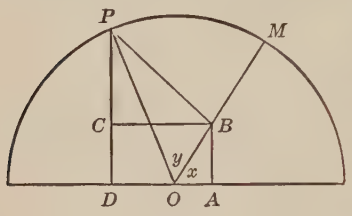


FIG. 110.

177. The Proof Continued. The above formulas are often called *addition theorems.* In the development of them, angles x and y were taken acute. However, the formulas hold for any angles. If in place of x we write $x' = 90° + x$, we obtain

$$\sin (x' + y) = \sin (90° + x + y) = \cos (x + y)$$
$$= \cos x \cos y - \sin x \sin y.$$

But

$$\cos x = \sin (90° + x) = \sin x',$$
$$\sin x = -\cos (90° + x) = -\cos x'.$$

Substituting these values, we find

$$\sin (x' + y) = \sin x' \cos y + \cos x' \sin y.$$

That is, (8) holds true if either angle is increased by 90°. It is therefore true for all positive angles, since any positive angle can be obtained by adding 90° enough times to an acute angle. The proof for the cosine is similar.

178. Sine and Cosine of the Difference of Two Angles. Since the preceding formulas (8) and (9) hold true for positive values of x and y of any magnitude whatever, they are true when in place of y one uses the notation $360° - y$. The equations (8) and (9) then become

$$\sin [x + (360° - y)]$$
$$= \sin x \cos (360° - y) + \cos x \sin (360° - y).$$

$$\cos [x + (360° - y)]$$
$$= \cos x \cos (360° - y) - \sin x \sin (360° - y).$$

But

$$\cos (360° - y) = \cos y, \quad \text{and} \quad \sin (360° - y) = -\sin y.$$

Moreover,

$$\sin [x + (360° - y)] = \sin [360° + (x - y)] = \sin (x - y),$$

since the initial line and the final line are the same for $360° + (x - y)$ and for $(x - y)$; and similarly

$$\cos [x + (360° - y)] = \cos (x - y).$$

Substituting these values and simplifying, we find

(10) $$\sin (x - y) = \sin x \cos y - \cos x \sin y.$$

(11) $$\cos (x - y) = \cos x \cos y + \sin x \sin y.$$

Applications. The above formulas (8), (9), (10), (11) are useful in solving equations, in calculating tables, in studying simple harmonic motion, and in simplifying formulas for logarithmic computation. So many theorems can be deduced from them that they are sometimes called *fundamental formulas*. They should be memorized.

Note. The first proof of the formula for $\sin (x + y)$ was given by Adrian Romanus (1561–1625), professor of mathematics and medicine at the University of Louvain. The formulas for $\sin (x - y)$, and $\cos (x + y)$ were given by Pitiscus in his trigonometry published in 1595.

EXERCISES

1. Putting $75° = 30° + 45°$, find $\sin 75°$, $\cos 75°$, and $\tan 75°$.

Suggestion.

$$\sin 75° = \sin (30° + 45°) = \sin 30° \cos 45° + \sin 45° \cos 30°$$
$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{1 + \sqrt{3}}{2\sqrt{2}} = .96593 \cdots.$$

Does this value check with the value given in the table of natural functions?

2. Could the above method be used in checking the values in any table of natural trigonometric functions?

3. A common mistake made by beginners is to assume $\sin (x + y) = \sin x + \sin y$. Test each of the following statements by means of the table of natural functions and see whether or not it is true.

(a) Is $\sin 90° = \sin (45° + 45°) = \sin 45° + \sin 45°$?

(b) Is $\sin 90° = \sin (30° + 60°) = \sin 30° + \sin 60°$?

4. Putting $90° = 30° + 60°$, prove $\sin 90° = 1$, $\cos 90° = 0$.

5. Putting $90° = 45° + 45°$, prove $\cos 90° = 0$, $\sin 90° = 1$.

6. Putting $15° = 45° - 30°$, find $\sin 15°$, $\cos 15°$.

7. Putting $60° = 30° + 30°$, find $\sin 60°$, $\cos 60°$.

8. Putting $0° = 60° - 60°$, find $\sin 0°$, $\cos 0°$.

9. Prove $\sin (90° + x) = \cos x$ by using the formula for $\sin (x + y)$.

10. Prove each of the following relations by means of the formulas for $\sin (x \pm y)$ and $\cos (x \pm y)$.

(a) $\sin (180° - x) = \sin x$.

(b) $\sin (180° + x) = -\sin x$.

(c) $\sin (360° - x) = -\sin x$.

(d) $\cos (180° - x) = -\cos x$.

(e) $\cos (180° + x) = -\cos x$.

(f) $\cos (360° - x) = \cos x$.

11. Find the value of $\cos (x + y)$, having given $\sin x = 3/5$ and $\sin y = 8/17$, where x and y are positive acute angles.

Solution. From Fig. 111 it will be seen that if $\sin x = 3/5$, then $\cos x = 4/5$. If $\sin y = 8/17$, then $\cos y = 15/17$. Substituting these values in the expanded form,

$$\cos (x + y) = \frac{4}{5} \times \frac{15}{17} - \frac{3}{5} \times \frac{8}{17} = \frac{36}{85}. \quad Ans.$$

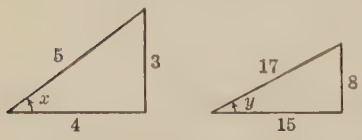


Fig. 111.

12. Find the values of $\sin (x + y)$, $\sin (x - y)$, $\cos (x - y)$, and $\cos (x + y)$, if $\sin x = 4/5$, and $\sin y = 12/13$, x and y being in the first quadrant.

13. At a horizontal distance of 36 feet from the center of the base on which a statue rests, the angles of elevation of the top and bottom of the statue are 40° and 31°, respectively. Find the height of the statue and the height of the base.

14. At a point A, the angle of elevation of a point B (the top of a vertical wall) is α, and the angle of depression of its base is β. Find the height of the wall, the horizontal distance AC being a feet.

SOLUTION.

(1) $DB = a(\tan \alpha + \tan \beta)$,

or

(2) $DB = \dfrac{a \sin (\alpha + \beta)}{\cos \alpha \cos \beta}$.

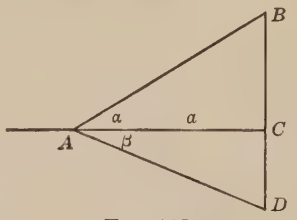


FIG. 112.

Equation (1) is adapted for solution by natural functions, while (2) is adapted to logarithmic computation.

15. Find the height of the wall in Ex. 14 if $\alpha = 36°$, $\beta = 18°$, and $a = 46$ ft.

16. At a point A on a horizontal plane the angle of elevation of the top of a crag is α, and at a point B, x feet farther away in the same horizontal plane, the angle of elevation is β. Find the height of the crag.

$$Ans. \quad \text{Height} = x \frac{\tan \alpha \tan \beta}{\tan \alpha - \tan \beta} = \frac{x \sin \alpha \sin \beta}{\sin (\alpha - \beta)}.$$

17. Solve Ex. 16 if $x = 90$ ft., $\alpha = 62°$, $\beta = 41°$.

18. From the top of a building of height h, a man observes that the angle of elevation of a tower is α. From the foot of the building, he finds that the angle of elevation is β. Show that the height of the tower exceeds the height of the building by

$$\frac{h \sin \alpha \cos \beta}{\sin (\beta - \alpha)}.$$

19. Find the height in Ex. 18 if $h = 18$ ft., $\alpha = 38°$, $\beta = 52°$.

20. A ravine is spanned by a horizontal bridge of length l. If the sides of the ravine make angles of α and β with the horizontal, show that the height of the bridge above the bottom of the ravine is

$$\frac{l}{\operatorname{ctn} \alpha + \operatorname{ctn} \beta} = \frac{l \sin \alpha \sin \beta}{\sin (\alpha + \beta)}.$$

21. Find the height in Ex. 20 if $l = 298$ ft., $\alpha = 54°$, $\beta = 32°$.

22. To find the height h of a hill above a point P in the horizontal plane at the foot of the hill, a pole of length d is set vertically on the

top of the hill and the angles T and β, the angles of elevation of the top and bottom of the pole, are measured. Prove

$$h = \frac{d \tan \beta}{\tan T - \tan \beta} = \frac{d \sin \beta \cos T}{\sin (T - \beta)}.$$

179. Tangent of the Sum and of the Difference of Two Angles. By (2), § 165, we have

$$\tan (x + y) = \frac{\sin (x + y)}{\cos (x + y)} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y}.$$

Dividing both numerator and denominator of the right-hand member by $\cos x \cos y$, we find

$$\tan (x + y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\sin y \cos x}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}}.$$

Hence

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \tag{12}$$

Similarly,

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \tag{13}$$

Formula (13) is used in finding the angle between two lines.

180. Angle between Two Lines. From the previous paragraph it is seen that if x and y are the angles which two given lines make with the x-axis (Fig. 113), and if α is the difference of these angles (that is, $\alpha = x - y$), then

$$\tan \alpha = \tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

If $\tan x$, the slope of one line, is represented by m_1, and $\tan y$, the slope of the other, is represented by m_2, then the tangent of the angle between the lines is given by the formula,

$$\tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2}, \tag{14}$$

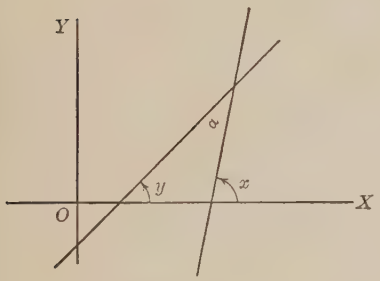


Fig. 113.

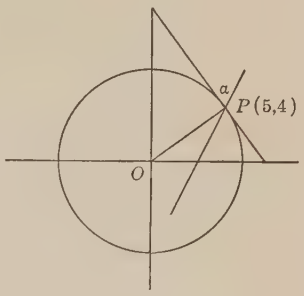


Fig. 114.

where m_1 denotes the slope of the line of greater inclination.

Example 1. Find the angle between the lines $y = 4x - 13$ and $y = 2x + 11$.

Solution. The slope $m_1 = 4$, and $m_2 = 2$. Hence,

$$\tan \alpha = \frac{4 - 2}{1 + 8} = \frac{2}{9} = 0.22222,$$

and

$$\alpha = 12^\circ\ 31'\ 40''.$$

Example 2. Find the angle between the line $y = 2x - 6$ and the circle $x^2 + y^2 = 41$, at the point of intersection (5, 4). (See Fig. 114.)

Solution. For the straight line, $m_2 = 2$. For the circle the slope of the radius OP is $4/5$; hence the slope of the tangent at P is $m_1 = -5/4$. Therefore,

$$\tan \alpha = \frac{-5/4 - 2}{1 - 2(5/4)} = \frac{13}{6} = 2.16667,$$

and the acute angle $\alpha = 65^\circ\ 14'$, nearly.

If two lines are *parallel*, they make equal angles with the x-axis. Let these angles be x and y. (See Fig. 115.) Then $\tan x = \tan y$. Hence $m_1 = m_2$. Compare § 97.

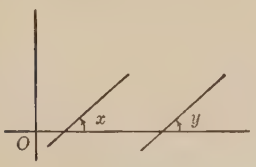


FIG. 115.

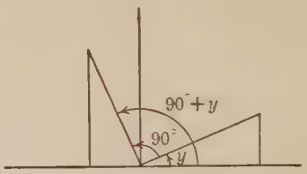


FIG. 116.

If two lines are *perpendicular* (Fig. 116), the angles they make with the x-axis must differ by 90°, that is, x must equal $90° + y$. Hence we have

$$\tan x = \tan (90° + y) = -\operatorname{ctn} y = \frac{-1}{\tan y}, \tag{15}$$

whence

$$m_1 = \frac{-1}{m_2}. \tag{16}$$

Compare § 98.

EXERCISES

1. Find the slope of each of the following lines.

(*a*) $y = 3x + 7$. (*c*) $y = 5x - 6$. (*e*) $x - 2y = 8$. (*g*) $5y = 4x + 5$.

(*b*) $x = 2y + 9$. (*d*) $y = 4x - 7$. (*f*) $\frac{x}{2} - \frac{y}{5} = 1$. (*h*) $2y = 5x - 3$.

2. Find the slope and the intercepts for each of the following lines. Draw the graphs.

(*a*) $3x - 5y = 15$.
(*b*) $7x + 3y = 21$.
(*c*) $2x - 5y = 11$.
(*d*) $3x - 7y = 4$.
(*e*) $x - 2y = 10$.
(*f*) $2x - 3y = 18$.
(*g*) $x - 5y = 5$.
(*h*) $2x - 5y = 10$.
(*i*) $2x + 5y = 20$.
(*j*) $y = 2x - 5$.
(*k*) $3x - 7y = 10$.
(*l*) $2y = 3x - 4$.
(*m*) $\frac{x}{6} + \frac{y}{5} = 1$.
(*n*) $\frac{x}{5} + \frac{y}{2} = 1$.

3. A timber in the frame of a bridge has a rise of 10 feet in a run of 12 feet. What is its slope? What is the slope of an iron rod perpendicular to the above timber?

4. Find the angle the line $3x - 4y = 12$ makes with the x-axis. With the y-axis.

5. Find the angle which the line

(*a*) $2x - y = 7$ makes with the line $y = x + 3$.

(*b*) $3x - y = 10$ makes with the line $y = 2x + 5$.

(*c*) $y = 4x + 10$ makes with the line $y = 2x - 4$.

(*d*) $y = 3x + 4$ makes with the line $y = 4$.

(*e*) $y = 2x + 7$ makes with the line $x = 2$.

(*f*) $y = -x + 1$ makes with the line through (2, 4) and (7, 10).

(*g*) $x - 2y = 10$ makes with the line through (0, 0) and (− 4, 8).

6. Find the equations of the lines satisfying the following conditions.

(*a*) through (2, 3) and parallel to $y = 5x + 8$.

(*b*) through (3, 5) and parallel to $y = \frac{2x}{3} - 7$.

(*c*) through (1, 6) and perpendicular to $y = 2x + 5$.

(*d*) through (2, 5) and perpendicular to $3x - 5y = 7$.

7. It is desired to find the angle between two intersecting lines on a blue-print. One line has a slope equal to 3, and the other a slope equal to 3/4. What is the angle?

8. Two straight railroads intersect in such a way that the slope of one with respect to a given line is 8/5 and the slope of the other with respect to the same line is 1/3. Find the angle between them.

9. The sides of a triangle have slopes equal to 1/2, 1, and 2, respectively. Prove that the triangle has two equal angles and is therefore isosceles.

10. Find tan 15°, using the relation $15° = 60° - 45°$.

11. Find tan 75°, using the relation $75° = 30° + 45°$.

12. Verify each of the following identities.

(*a*) $\tan (45° + x) = \dfrac{1 + \tan x}{1 - \tan x}$.

(*b*) $\tan (60° + x) = \dfrac{\sqrt{3} + \tan x}{1 - \sqrt{3} \tan x}$.

(*c*) $\tan x + \tan y = \dfrac{\sin (x + y)}{\cos x \cos y}$.

(*d*) $\tan (45° - x) = \dfrac{1 - \tan x}{1 + \tan x}$.

13. Find the angles of a triangle whose vertices are (1, 2), (8, 5) and (5, 10).

14. Find the angles of a triangle whose vertices are (5, 4), (7, 4) and (7, 10).

181. Functions of Double an Angle. Since the formulas for the functions of the sum of two angles are true for all values of x and y, they are true when $y = x$.

(*a*) In the equation

$$\sin (x + y) = \sin x \cos y + \sin y \cos x,$$

let $y = x$. Then

$$\sin 2x = \sin x \cos x + \sin x \cos x,$$

or

$$\sin 2x = 2 \sin x \cos x. \tag{17}$$

The sine of any angle is equal to twice the product of the sine of half the angle times the cosine of half the angle. Thus

$$\sin 48° = 2 \sin 24° \cos 24°.$$

(*b*) Similarly

$$\cos 2x = \cos x \cos x - \sin x \sin x$$

or

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{18}$$

Since $\cos^2 x = 1 - \sin^2 x$, substituting in (18), we have

$$\cos 2x = 1 - 2 \sin^2 x. \tag{19}$$

Since $\sin^2 x = 1 - \cos^2 x$, substituting in (18), we have

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$

or

$$\cos 2x = 2 \cos^2 x - 1. \tag{20}$$

(*c*) Likewise, since

$$\tan 2x = \frac{\tan x + \tan x}{1 - \tan x \tan x},$$

we may write

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \tag{21}$$

By means of these formulas the functions of twice an angle are easily computed, provided the functions of the angle itself are known.

182. Functions of Half an Angle. It is sometimes necessary to express the functions of half an angle in terms of the whole angle. This is most easiiy accomplished by means of (19) and (20), § 181. From (19), § 181, we have

$$2 \sin^2 x = 1 - \cos 2x,$$

or

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}.$$

Since these formulas hold for any value of the angle, x may be replaced by $x/2$. Hence we may write

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}. \tag{22}$$

The sign to be used depends upon the quadrant in which the angle $x/2$ is situated. For example, for any acute angle, such as $22\frac{1}{2}°$, the sine is positive, and we must write

$$\sin 22\tfrac{1}{2}° = \sqrt{\frac{1 - \cos 45°}{2}}.$$

The sign in (22) is thus plus if $x/2$ is less than 180°, that is, if x is less than 360°. If x is greater than 360° but less than 720°, the sign in (22) is minus. Similarly, from (20), § 181, we have

$$2 \cos^2 x = 1 + \cos 2x,$$

or

$$\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}.$$

Putting $x/2$ for x, we may write

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}. \tag{23}$$

The sign is determined by the sign of cos $(x/2)$, which is plus if $x/2$ is less than 90°, that is, if x is less than 180°. Dividing (22) by (23), we get

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}. \tag{24}$$

The sign of the radical is to be taken plus if $x/2$ is less than 90°, that is, if x is less than 180°.

NOTE. Formulas (22), (23) and (24) have many important applications in mathematics. If the functions of an angle are known, the functions of an angle half as large are readily computed.

EXERCISES

1. Given $\sin 30° = 1/2$, $\cos 30° = \sqrt{3}/2$, $\tan 30° = 1/\sqrt{3}$; find sin 60°, cos 60°, tan 60°.

2. Given the functions of 60°, find the functions of 120°.

3. Given the functions of 45°, find the functions of 90°.

4. Given the functions of 60°, find the functions of 30°.

5. Given the functions of 90°, find the functions of 45°.

6. Given the functions of 45°, find the functions of $22\frac{1}{2}$°.

7. Given the functions of 30°, find the functions of 15°.

8. A tower is 54 feet high. The lower 24 feet is painted black, the upper 30 feet white. At what horizontal distance from the base of the tower will the black portion subtend the same visual angle as the white portion?

9. A right triangle has a hypotenuse 30 feet long. Show that the area is $A = 225 \sin 2\alpha$, where α is the inclination of either leg to the hypotenuse. For what value of α will the area be greatest?

10. Telephone wires h feet above the ground exert a horizontal pull of p pounds on each pole. The end pole is held in position by a guy wire l feet long and with an inclination θ. (See Fig. 117.) If F is the tension of the guy wire, show that for a perfect balance

$$ph = Fl \sin \theta \cos \theta = \tfrac{1}{2}Fl \sin 2\theta.$$

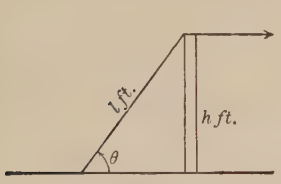


FIG. 117.

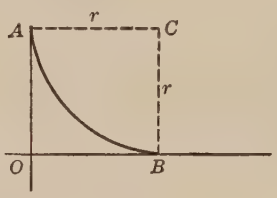


FIG. 118.

11. Two railroads meet at right angles at O. They are connected by a quadrant of a circle 2400 feet long. (See Fig. 118.) What is the distance of either point A or B from O?

183. Sums and Differences of Sines Expressed as Products. From (8) and (10), we have

$$\sin (x + y) = \sin x \cos y + \sin y \cos x.$$
$$\sin (x - y) = \sin x \cos y - \sin y \cos x.$$

By addition, we find

(25) $$\sin (x + y) + \sin (x - y) = 2 \sin x \cos y.$$

By subtraction, we get

(26) $$\sin (x + y) - \sin (x - y) = 2 \sin y \cos x.$$

Let $x + y = A$, and $x - y = B$. Then by solving for x and y we find

$$x = \frac{A + B}{2}, \quad \text{and} \quad y = \frac{A - B}{2}.$$

Substituting the values of $x + y$, $x - y$, x, y, in terms of

A and B in (25) and (26), we obtain

$$\text{(27)}\quad \sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

$$\text{(28)}\quad \sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right).$$

NOTE. The principal use of these formulas is in making certain transformations in integral calculus. A minor use is in adapting certain formulas to logarithmic work by replacing sums and differences by products.

Dividing (27) by (28) member for member, we obtain

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}}{2 \sin \dfrac{A-B}{2} \cos \dfrac{A+B}{2}}.$$

Hence,

$$\text{(29)}\qquad \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

184. Law of Tangents. *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of their opposite angles is to the tangent of half their difference.*

Denote by D the common value of the fractions in the law of sines, that is,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = D.$$

Then $a = D \sin A$, $b = D \sin B$, $c = D \sin C$, and

$$\text{(30)}\quad \frac{a+b}{a-b} = \frac{D \sin A + D \sin B}{D \sin A - D \sin B} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

Equating values in (29) and (30), we get

$$\text{(31)}\qquad \frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

Solving for $\tan \frac{1}{2}(A - B)$, we get

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B). \tag{32}$$

If two sides and the included angle are known, as a, b, and C, the law of tangents may be employed in finding the two unknown angles A and B, for the values $a + b$, $a - b$, $A + B = 180° - C$, and $\tan \frac{1}{2}(A + B)$ are readily obtained. Similarly, we get

$$\frac{a + c}{a - c} = \frac{\tan \frac{1}{2}(A + C)}{\tan \frac{1}{2}(A - C)}, \tag{33}$$

and

$$\frac{b + c}{b - c} = \frac{\tan \frac{1}{2}(B + C)}{\tan \frac{1}{2}(B - C)}. \tag{34}$$

NOTE. A check formula which involves all the sides and all the angles of a triangle was devised by Sir Isaac Newton and was given in his *Arithmetica Universalis* in 1707. It is

$$\frac{a + b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}. \tag{35}$$

If logarithms are used in solving triangles where two sides and the included angle are given, the law of tangents, which involves products, is preferred to the law of cosines, which involves sums. Even when logarithms are not used, the law of tangents is convenient.

EXAMPLE. If $a = 54.7$, $b = 42.6$, $C = 58°$, find A, B, and c.

SOLUTION. $a + b = 97.3$, $a - b = 12.1$, $A + B = 180° - C = 122°$, and $\frac{1}{2}(A + B) = 61°$. Then

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B)$$

$$= \frac{12.1}{97.3} \times 1.8040 = .2243, \text{ nearly.}$$

Therefore $\frac{1}{2}(A - B) = 12° 39'$ nearly. Adding this result to $\frac{1}{2}(A + B) = 61°$, we get $A = 73° 39'$. Similarly subtracting from $\frac{1}{2}(A + B) = 61°$, we find $B = 48° 21'$.

To find side c, by the law of sines, we have

$$c = \frac{a \sin C}{\sin A} = \frac{54.7 \sin 58°}{\sin 73° 39'} = 48.34$$

Checking by (35), we find

$$\frac{97.3}{48.34} = \frac{.9757}{.4848}, \text{ nearly.}$$

185. Formulas for Solving Oblique Triangles. The law of sines, the law of tangents, and the formulas for the area of a triangle, are well adapted to logarithmic computation since they involve products, quotients and roots. In the law of cosines, which involves sums and differences, the labor-saving advantages of logarithmic computation are not so marked.

The following classification is convenient.

I. When two angles and a side are given, use the law of sines.

II. When two sides and an angle opposite one of them are given, use the law of sines.

III. When two sides and the included angle are given, use (*a*) the law of cosines, or (*b*) the law of tangents.

IV. When all the sides are given, use the law of cosines.

NOTE. The essential principle of the law of tangents was known to Ptolemy about 150 A.D. In its present form it was first given by Vieta, and it was improved by Thomas Finck (1561–1656), a Danish physician and mathematician.

EXERCISES

1. Having given $a = 30$, $b = 20$, $C = 52°$, solve for A and B.
2. Having given $a = 45$, $b = 40$, $C = 64°$, solve for A and B.
3. Having given $a = 50$, $c = 40$, $B = 62°$, solve for A and C.
4. Having given $a = 10$, $b = 12$, $C = 56°$, solve the triangle.
5. Having given $b = 25$, $c = 30$, $A = 70°$, solve the triangle.
6. Solve the triangle having given $a = 84.2$, $b = 70.8$, $C = 64°$.
7. Solve the triangle having given $b = 67.2$, $c = 74.6$, $A = 56°$.

8. Solve the triangle having given $a = 104.3$, $c = 76.1$, $B = 62°$.

9. At the time of a given observation the distances of the earth and Mars from the sun were, respectively, 92 000 000 miles and 141 000 000 miles. The angle ESM was 72°. Find EM, the distance from the earth to Mars.

10. Two mountain peaks are, respectively, 17 miles and 25 miles from the point of observation. The angle between their directions from the point of observation is 48°. Find their distance apart.

11. Two stars A and B are distant 4.2×10^{15} and 5.4×10^{16} miles from the earth. The angle between their directions is 12°; find their distance apart.

12. Using the law of tangents, prove that an equilateral triangle is also equiangular.

13. Using the law of tangents, prove that an equiangular triangle is also equilateral.

14. A timbered valley is roughly in the form of a triangle. The vertices A, B, and C are marked by high peaks. If AB is 10 miles, AC 8 miles, and angle A 65°, solve for the remaining side and angles. Find the area.

15. In a survey of a triangular area, a part of the data was lost. It is known that $A = 62° 14'$, $B = 78° 27'$ and the sum of the sides a and b is 1920 feet. Is the data sufficient to solve the triangle?

Law of Tangents

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A+B)$$

Law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

CHAPTER XII

POLAR COORDINATES AND ALLIED TOPICS

186. Introduction. The method of locating a house, town, or other object by giving its distance and direction is extensively used. For instance, a mountain may be 10 miles distant in a direction E 22° N. A ship is distant 8 miles in a direction E 27° S.

187. Locating Points. A fixed point O is chosen as the origin and called the ***pole,*** and a fixed line OX is taken as the polar axis. Any point P in the plane is definitely located as soon as the distance OP and the angle XOP are known. OP is called the ***radius vector*** and is denoted by r or by ρ (rho). The angle XOP is called the ***vectorial angle*** and is denoted by θ (theta). The numbers (ρ, θ) are called the ***polar coordinates*** of P.

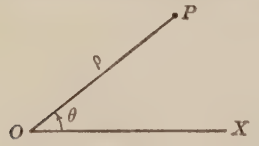


Fig. 119.

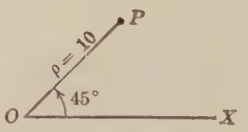


Fig. 120.

188. Definition Extended. The vectorial angle θ may be positive or negative, as in trigonometry. The radius vector ρ is considered positive if P lies on the terminal line of θ, and negative if P lies on that line produced through the pole O. Thus the point P may be located by $\rho = 10$ and $\theta = 45°$, that is, $(10, 45°)$. The same point is located by $\rho = -10$ and $\theta = 225°$, that is, $(-10, 225°)$.

189. Use of ρ and θ in Computation, Angle Small. In § 66, it was shown that the length of an arc s is given by

the equation

$$(1) \qquad s = r\theta \text{ (radians)} = 4.85 \times 10^{-6} r\theta''.$$

The distance to an object is given by

$$(2) \qquad r = \frac{s}{\theta \text{ (radians)}} = 206265 \frac{s}{\theta''}.$$

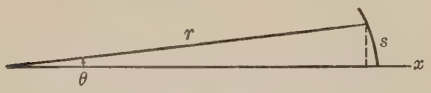


FIG. 121.

If the angle θ is less than 2°, its radian measure is very nearly equal to the sine or tangent of the angle. To illustrate from the tables:

$$\sin 2° = 0.03490, \qquad \tan 2° = 0.03492,$$
$$2° = 0.03491 \text{ (radian)}.$$

Since these values are the same to four decimal places, the radian measure of angle θ, used in place of $\sin\theta$ or $\tan\theta$, is an approximation frequently made.

EXAMPLE 1. To find log tan 2′ 54″.

SOLUTION. $2' 54'' = 0.048333° = (0.048333)(0.017453 \text{ radian})$.

$$\log 0.048333 = 8.68425 - 10$$
$$\log 0.017453 = 8.24187 - 10$$
$$\log \tan 2' 54'' = 6.92612 - 10,$$

which is correct to 5 decimal places.

EXAMPLE 2. A railroad has a rise of 20 feet in a distance of 2240 feet along the track. What angle does it make with the horizontal?

SOLUTION. $\sin A = \frac{20}{2240}$ = number of radians in A, approximately.

Hence $$A = \frac{20}{2240} \times 206265'' = 1841.6'' = 30' 41.6''.$$

190. Area of a Circular Sector. The area of a circular sector is equal to the area of a triangle whose base is equal in length to the arc of the sector, and whose altitude is equal to the radius (Fig. 122); that is,

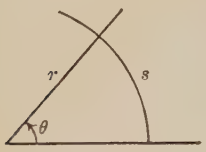


Fig. 122.

$$A = \tfrac{1}{2}rs. \tag{3}$$

But $s = r\theta$, θ being expressed in radians. Hence

$$A = \tfrac{1}{2}r \times r\theta = \tfrac{1}{2}r^2\theta. \tag{4}$$

Example. Find the area of a circular sector bounded by an arc 10 feet in length, the radius of the circle being 50 feet.

Solution. The radian measure of the angle $\theta = 10/50$. Hence

$$A = \tfrac{1}{2} \times 50^2 \times \frac{10}{50} = 250 \text{ sq. ft.}$$

EXERCISES

Plot the following points:

1. (5, 30°), (6, 0°), (10, 45°), (8, 60°), (12, 90°), (7, 180°).

2. (3, 120°), (2, 360°), (3, − 30°), (− 6, 60°), (− 4, 225°).

3. $(4, \pi/4)$, $(6, \pi/2)$, $(8, \pi)$, $(10, 3\pi/2)$, $(4, 2\pi)$.

4. Plot the following points. Express the position of each point by coordinates in which the radius vector is positive. (− 4, 60°), (− 6, 225°), (− 10, − 45°), (− 5, 30°).

5. Show that (ρ, θ) is symmetrical to:

(*a*) $(-\rho, \theta)$ with respect to the pole.

(*b*) $(\rho, -\theta)$ with respect to the polar axis.

6. The polar coordinates of a mountain peak are (24, 31°). If the angle θ is measured from the east toward the north, find its distance (*a*) east, (*b*) north.

7. A mountain 30 miles distant has an angle of elevation of .06 radians. Find its height above the point of observation.

8. A circular railroad curve is 800 feet long, and the radius is 960 feet. What is the angle between the tangents drawn at the extremities of the arc?

9. A point P is in 18° north latitude. A point Q is known to be directly north of P at a distance of 800 miles measured along the surface of the earth. Find the latitude of Q, if $r = 3960$ miles.

10. A sector of a circle has an arc of 5 inches and a radius of 11 inches. Find the area of the sector.

11. A circular sector, radius 10 inches, has a central angle of 114.6°. Find its area.

12. A sidewalk has a rise of 8 ft. in a distance along the sidewalk of 459 ft. What angle does it make with the horizontal?

13. How large a target at a distance of 30 feet will subtend the same angle at the eye as a target 4 feet in diameter at a distance of 1000 yards? What angle do they subtend?

14. A hill subtends an angle of $1\frac{1}{2}°$ at a horizontal distance of 4 miles. Find its height in feet.

15. During an eclipse of the sun a solar prominence was observed to subtend an angle of 15′ of arc. Find its height in miles, if the distance to the sun is 92 000 000 miles.

16. How far away is Saturn when its diameter of 74 000 miles subtends an angle of 18 seconds of arc?

17. Find the distance of Jupiter when its diameter of 88 000 miles subtends at the earth an angle of 37.5 seconds of arc.

18. Find the diameter of a sun spot in the center of the disc if it subtends an angle of 7″ of arc, when the sun is 92 700 000 miles away.

19. When the moon is 240 000 miles away, find the diameter of a crater on the moon, if it subtends an angle of 18″.

20. A railroad curve is the arc of a circle. If the angle at the center of the circle is 31° and the radius is 920 feet, find the length of the arc.

191. Length of a Line. The length of the line joining two points $P_1(\rho_1, \theta_1)$ and $P_2(\rho_2, \theta_2)$ is given by the cosine law:

$$l^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2). \tag{5}$$

This may be proved independently by substituting $\rho \cos \theta$ for x and $\rho \sin \theta$ for y in the corresponding formula for rectangular coordinates. (See § 192.)

192. Relation of Rectangular and Polar Coordinates. Let P be any point in the plane which has the rectangular

coordinates (x, y) and the polar coordinates (ρ, θ), the origin and pole being the same point and the x-axis and the polar axis coincident. Then

$$x = \rho \cos \theta, \quad \text{and} \quad y = \rho \sin \theta. \tag{6}$$

Hence, if the equation of the curve in rectangular coordinates is known, the equation of the curve in polar coordinates may be found by making the above substitution.

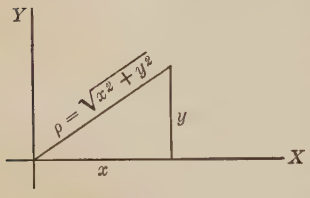


FIG. 123.

EXAMPLE. In rectangular coordinates, the equation of a circle whose center is at $(r, 0)$ and whose radius is r is

$$(x - r)^2 + y^2 = r^2 \quad \text{or} \quad x^2 + y^2 - 2rx = 0.$$

Substituting $\rho \cos \theta$ for x, and $\rho \sin \theta$ for y, we obtain

$$\rho^2(\sin^2 \theta + \cos^2 \theta) - 2r\rho \cos \theta = 0.$$

Since $\sin^2 \theta + \cos^2 \theta = 1$, this reduces to

$$\rho^2 - 2r\rho \cos \theta = 0.$$

Dividing the equation by ρ and transposing, we find

$$\rho = 2r \cos \theta.$$

It may be seen from Fig. 126 that this equation is correct.

To change an equation from polar to rectangular coordinates, the following substitutions are made.

$$\rho = \sqrt{x^2 + y^2}, \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}, \tag{7}$$

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \tan \theta = \frac{y}{x}.$$

EXERCISES

1. Find the distance between the points (20, 40°) and (30, 24°).
2. Find the length of the line joining (30, 10°) and (40, 28°).

3. A golf ball driven from the pole for a flag at (300, 0°) came to rest at (280, 15°). How far was it from the flag?

4. The position of a comet traveling in a plane curve is given by (100, 14°). Twenty-four hours later its position is (102.6, 18°). Find its approximate velocity in miles per second, ρ being given in millions of miles.

5. At a given time the positions along the plane of the ecliptic of the following planets were: Mercury (41, 0°); Venus (68, 24°); Earth (92, 57°). Estimate the distances between these planets from a graph, and check the results by calculation. The values of ρ are given in millions of miles.

6. A pumping station is located at a distance of 6 miles in a direction E 30° N. How far east is it? How far north?

7. A pendulum 48 inches long swings through an arc of 22°. Find the rectangular coordinates of its positions of instantaneous rest.

8. If $x = 9$ and $\rho = 15$, find y and θ. Is there more than one point which satisfies these conditions?

9. Find the polar coordinates of points whose rectangular coordinates are (*a*) (3, 4); (*b*) (12, 5); (*c*) (8, 6).

10. The polar coordinates of a point are (12, 30°). Find the rectangular coordinates of the point.

11. The base of an equilateral triangle whose side is 6 inches is taken as the polar axis. If one vertex is at the pole, find the coordinates of the other vertices.

12. Change the following from rectangular to polar coordinates. Discuss the locus in each case.

(*a*) $y = x/\sqrt{3}$.	*Ans.* $\theta = 30°$.
(*b*) $x - y = 0$.	*Ans.* $\theta = 45°$.
(*c*) $y = 5$.	*Ans.* $\rho \sin \theta = 5$.
(*d*) $x = 8$.	*Ans.* $\rho \cos \theta = 8$.
(*e*) $x^2 + y^2 = 36$.	*Ans.* $\rho = 6$.
(*f*) $x^2 + y^2 - 6x = 0$.	*Ans.* $\rho = 6 \cos \theta$.
(*g*) $x^2 + y^2 - 8y = 0$.	*Ans.* $\rho = 8 \sin \theta$.

13. Change the following equations from polar to rectangular coordinates. Discuss the locus in each case.

(*a*) $\theta = 30°$.	(*c*) $\rho \sin \theta = 4$.	(*e*) $\rho = 6 \sin \theta$.
(*b*) $\rho = 5$.	(*d*) $\rho \cos \theta = 10$.	(*f*) $\rho = 4 \cos \theta$.

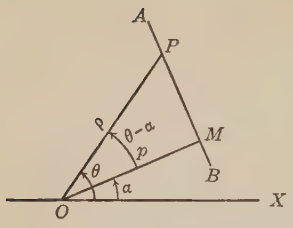


Fig. 124.

193. Polar Equation of a Straight Line. Let OX be the initial line, O the pole and AB any straight line. Let p be the length of OM, the perpendicular from O upon AB, XOM the angle α, and $P(\rho, \theta)$ any point on AB. Then from the right triangle OMP,

$$\frac{p}{\rho} = \cos(\theta - \alpha).$$

Hence

(8) $$\rho \cos(\theta - \alpha) = p,$$

which is the polar equation of the straight line AB. If $\alpha = 0°$, the line AB is perpendicular to the polar axis OX and the equation reduces to

(9) $\rho \cos\theta = p$, that is, $x = p$. (Fig. 125.)

If $\alpha = 90°$, the line AB is parallel to the polar axis. Since $\cos(\theta - 90°) = \sin\theta$, the equation is

(10) $\rho \sin\theta = p$, that is, $y = p$.

For a straight line passing through the pole and making an angle θ with the polar axis the equation is evidently

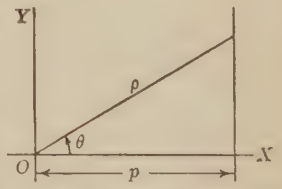


Fig. 125.

(11) $$\theta = c, \quad \text{a constant.}$$

194. Polar Equation of the Circle. The general form of the equation of the circle is not often used. Some special forms only will be considered.

(*a*) *The locus of* $\rho = r$. Since every point on a circle of radius r is at a distance r from the center, the equation of a circle with center at the pole and radius r is $\rho = r$.

(*b*) *The locus of* $\rho = 2r \cos \theta$.

This equation represents a circle of radius r which passes through the origin or "pole" and has its center on the polar axis at $(r, 0°)$. See Fig. 126, and the Example of § 192.

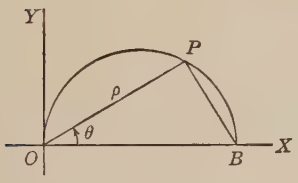


Fig. 126.

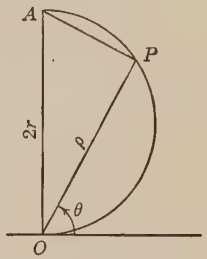


Fig. 127.

Proof. Let $P(\rho, \theta)$ be any point on the circle. Then $OP = \rho$ and angle $OPB = 90°$. Why? Then in the right triangle OPB,

$$\frac{\rho}{2r} = \cos \theta,$$

or

$$\rho = 2r \cos \theta. \tag{12}$$

(*c*) *The locus of* $\rho = 2r \sin \theta$.

This equation represents a circle of radius r which passes through the pole and has its center at $(r, \pi/2)$. It is tangent to the polar axis at O.

Proof. $OPA = 90°$. Why? Hence

$$\frac{\rho}{2r} = \cos (90° - \theta),$$

that is,

$$\rho = 2r \sin \theta. \tag{13}$$

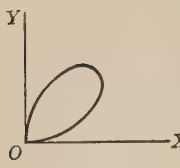


FIG. 128.

195. Graphs in Polar Coordinates. The graph of an equation in polar coordinates may be plotted in a way similar to that used in rectangular coordinates. A table of values of (ρ, θ) is formed. Each pair of values is then plotted and a smooth curve is drawn through these points.

EXAMPLE. Plot the graph of

$$\rho = 5 \sin 2\theta.$$

θ	0°	15°	30°	45°	60°	75°	90°
ρ	0	2.5	4.3	5	4.3	2.5	0

There is a loop in each of the four quadrants. It is left to the student to complete the graph. This figure is sometimes called the four-leaved rose.

EXERCISES

1. What is the locus of all points for which $\rho = 5$? What is the equation in polar coordinates of a circle whose center is at the origin and whose radius is 7?

2. What is the locus of all points for which $\theta = 20°$? $\theta = -20°$? What is the equation of a line passing through the origin and making an angle of 28° with the polar axis?

3. Prove that the equation of a line parallel to the polar axis and distant a units from it is $\rho \sin \theta = a$.

4. Where are all the points for which ρ is a given constant?

5. Where are all the points for which θ is a given constant?

6. What exceptions are there to the statement, "Every point has four distinct sets of coordinates with θ less in absolute value than 360°"?

7. Find the polar equation of the straight line for which $\alpha = 22°$ and $p = 4$.

8. The perpendicular from the pole upon a line is 7 units long and makes an angle of 36° with the initial line. Write the polar equation of the line.

9. Plot the locus of each of the following equations.

(*a*) $\rho = 4$. (*c*) $\rho = 6 \cos \theta$. (*e*) $\rho = 10 \cos 2\theta$.
(*b*) $\theta = 45°$. (*d*) $\rho = 6 \sin \theta$. (*f*) $\rho = 10 \sin 2\theta$.

10. The general equation of the straight line in rectangular coordinates is

$$Ax + By + C = 0.$$

Show that the polar form is

$$\rho(A \cos \theta + B \sin \theta) + C = 0,$$

and that for the special cases $A = 0$ (line parallel to x-axis) or $B = 0$ (line parallel to y-axis), the equation has the forms $\rho \sin \theta = a$, and $\rho \cos \theta = b$, respectively.

11. Show that the general equation of the circle

$$x^2 + y^2 + Ax + By + C = 0$$

can be expressed in polar coordinates in the form:

$$\rho^2 + \rho(A \cos \theta + B \sin \theta) + C = 0.$$

196. Components of a Force. Resultant. In physics and in engineering a line-segment is often used to represent velocity, an acceleration, or the magnitude of a force. Thus a force of 20 lb. acting in a direction E 45° N may be represented by a line 20 inches long, extending in a north-easterly direction. If the force is R and its inclination to the x-axis is α, then (Fig. 129)

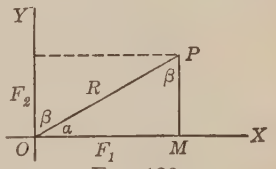


FIG. 129.

$$(14) \qquad F_1 = R \cos \alpha; \quad F_2 = R \sin \alpha = R \cos \beta.$$

These equations (14) are essentially the same as equations (6), § 192. More generally, two forces F_1 and F_2 (not necessarily perpendicular to each other), which together would be equivalent to a single force R, are called the ***components*** of R; R is called the ***resultant*** of the forces F_1 and F_2.

If the two component forces are mutually perpendicular, the component force in any direction is equal to the magnitude of the given force multiplied by the cosine of the included angle.

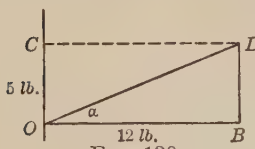


FIG. 130.

EXAMPLE. Two forces of 5 and 12 lb., respectively, act at right angles to each other. Find the resultant and the angle which the resultant makes with the second force. Represent the forces by OC and OB and complete the parallelogram. OD is the resultant.

$$OD = \sqrt{OB^2 + OC^2} = \sqrt{12^2 + 5^2} = 13.$$

$$\cos \alpha = \frac{OB}{OD} = \frac{12}{13} = .9231, \qquad \alpha = 22^\circ\ 37' \text{ nearly.}$$

197. The Inclined Plane. If the weight W of an object which rests on a plane inclined at an angle α to the horizontal is represented by the vertical line BA, Fig. 131, then the pressure on the plane can be represented by BC, and the force down the plane by CA. Hence $BA = W$, the vertical force; and we have

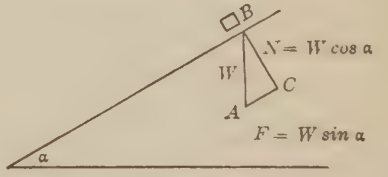


FIG. 131.

$$(15)\quad \begin{cases} BC = W \cos \alpha, \text{ the force perpendicular to the plane.} \\ CA = W \sin \alpha, \text{ the force parallel to the plane.} \end{cases}$$

These equations are essentially the same as equations (6), § 192.

EXAMPLE. A block weighing 82 lb. rests on a smooth plane inclined at an angle of 28°. Find the components of the force (*a*) down the plane, (*b*) perpendicular to the plane.

SOLUTION. $W = 82$ lb., $\alpha = 28^\circ$.

Hence $F = 82 \sin 28^\circ = 38.5$ lb.

$N = 82 \cos 28^\circ = 72.4$ lb.

198. The Screw. The screw may be considered as a modified form of the inclined plane. The screw is used, for example, to transmit vertical pressure when force is applied horizontally at the end of a lever of length l. When the lever arm is given one complete rotation ($2\pi l$) around the screw, the screw is turned so as to be raised by the distance d (the distance between adjacent threads in an ordinary screw), called the pitch of the screw. Hence if W is the weight raised, and F the force applied, we have by equating the work done (since work equals force times distance)

$$Wd = 2\pi lF.$$

199. Composition of Forces. The composition of two forces f_1 and f_2 into a third force f_3, equivalent to the sum of the other two forces, requires a three-sided polygon. It is often called the triangle of forces. If we choose to represent the component forces f_1 and f_2 by lines AB and AB' drawn from the same point, we get the parallelogram of forces, which is a convenient way of solving such problems. The resultant force is the diagonal of the parallelogram.

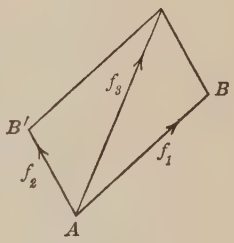


FIG. 132.

The conditions for equilibrium may be stated as follows:

Three forces, which, when simultaneously applied at a point, balance each other, can be correctly represented in magnitude and in direction by the three directed sides of a triangle taken so as to form a complete circuit, and conversely.

EXERCISES

1. Find the horizontal and vertical projections of each of the following line segments.

(*a*) Length 22 ft., making an angle of 28° with the horizontal.

(*b*) Length 34 ft., making an angle of 52° with the horizontal.

2. A car is drawn by means of a cable. If a force of 1200 lb. exerted along the track is required to pull the car, find the pull along the cable when the cable makes an angle of 18° with the track.

3. The draft (horizontal pull) of a plow is 360 lb. What power must be supplied to operate it if the traces of the horses make an angle of 18° with the horizontal?

4. A horse pulls on a towline with a force of 360 lbs. The towline makes an angle of 12° with the direction of the canal. Find the effective force which drives the boat forward. With what force is the boat being pulled toward the shore? Discuss the effect of using a short towline.

5. A roof 30 ft. long and 18 ft. wide is inclined at an angle of 26°. How many square feet will it shelter from rain falling vertically?

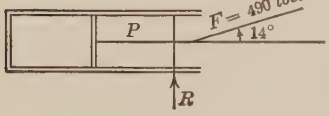


FIG. 133.

6. In the adjoining figure, the reaction R of the guide is vertical, P is horizontal. Find R and P, if $F = 490$ lb.

7. A roof is inclined at an angle of 36°. The wind strikes it horizontally with a force of 2000 lb. Find the pressure perpendicular to the roof.

8. An automobile weighing 2800 lb. stands on a pavement inclined at 12°. What force must be overcome by the brakes to prevent it rolling down hill, assuming that there is no rolling friction?

9. Two men, working on a capstan, walk in a circle 8 feet in diameter and each exerts a force of 50 lb. With each complete turn π feet of rope are pulled in. What is the pull along the rope, neglecting friction?

10. A house that is being raised by jackscrews exerts a pressure of 2800 lb. on each of them. If the pitch of the screw is 5/16 in. and it is operated by a lever 30 in. long, what force must be applied at the end of the lever to turn the screw?

11. A farm stretches as an inclined plane upon a hillside of 16° inclination. The actual surface area of the farm is 52 acres. Compute the area of the projection of this farm upon a horizontal plane.

12. A box weighing 56 lb. rests on a rough horizontal board. It takes a horizontal force of 12 lb. to move it. Through what angle must the board be inclined so that the box will be on the point of sliding down the plane?

13. A rifle ball having a muzzle velocity of 1296 ft. per second is fired at an inclination of 38°. Find the horizontal and vertical components of its velocity.

14. The horizontal distance between the two extreme positions of a pendulum 39.1 in. long is 5.74 inches. Through what angle does it swing? At the end of each swing, how far is the pendulum above its lowest position?

15. Draw a triangle ABC whose sides AB, BC, and CA are 7, 9, and 11 units long, respectively. If ABC is a triangle for three forces in equilibrium, and side BC represents a force of 28 lb., acting in the direction BC, show in the diagram the directions of the other forces and find their magnitudes.

16. A child weighing 72 lb. is sitting in a swing, the seat being 20 ft. from the point of support. (*a*) What horizontal force would be required to hold the child 12 feet to one side of the vertical through the point of support? (*b*) What would be the tension of the rope? Solve graphically.

17. A circle two feet in radius is projected on a plane inclined 18° to the plane of the circle. Find the area of the projection.

18. Find the resultant of two forces of 8 and 12 lb., respectively, acting at an angle of 60°.

19. Two forces of 30 and 40 lb., respectively, act at a point, and make an angle of 120° with each other. Find their resultant.

20. Draw two lines, AB and AC, including an angle of 150° and suppose a force of 20 units acts from A to B, and a force of 25 units from A to C. Find the resultant of the forces and the number of degrees in the angle which the resultant makes with AB.

21. Show that forces of 3, 4, and 8 lb. could not keep a particle in equilibrium.

22. Two forces, one of 200 lb. and the other of 150 lb., make an angle of 56° 28′. Find the magnitude and direction of their resultant.

200. Normal Form of the Equation of a Straight Line. The equation (8), § 193, for a straight line in polar coordinates,

$$\rho \cos (\theta - \alpha) = p,$$

leads to an important form in rectangular coordinates. Expanding $\cos (\theta - \alpha)$, we may write this equation in

the form

$$\rho \cos \theta \cos \alpha + \rho \sin \theta \sin \alpha = p,$$

or

$$x \cos \alpha + y \sin \alpha = p, \tag{16}$$

which is called the ***normal form*** of the equation of a straight

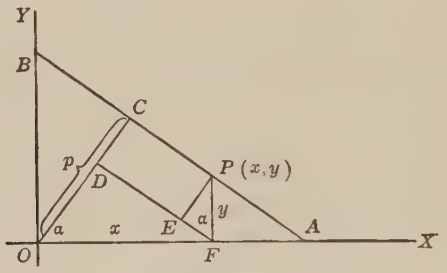


Fig. 134.

line. The correctness of this equation can be seen also directly from Fig. 134, in which AB is the given line.

201. Reduction of an Equation to the Normal Form. The general equation

$$Ax + By + C = 0 \tag{17}$$

is equivalent to (16) if the coefficients in the two equations are proportional. Multiplying both sides of (17) by k, and comparing coefficients with (16), we have

$$kA = \cos \alpha, \quad kB = \sin \alpha, \quad kC = -p. \tag{18}$$

Squaring and adding the first two of (18), we find

$$k^2(A^2 + B^2) = (\sin^2 \alpha + \cos^2 \alpha) = 1,$$

whence, solving for k, we get

$$k = \pm \frac{1}{\sqrt{A^2 + B^2}}.$$

Multiplying (17) by this value of k, we obtain

$$(19) \qquad \frac{Ax}{\pm\sqrt{A^2+B^2}} + \frac{By}{\pm\sqrt{A^2+B^2}} + \frac{C}{\pm\sqrt{A^2+B^2}} = 0.$$

The sign in the denominator must be taken opposite to the sign of C in order that $-p = \pm C/\sqrt{A^2+B^2}$ shall be negative.

Hence the equation of a line is reduced to the normal form by dividing the general equation of the line by the square root of the sum of the squares of the coefficients of x and y. If $C = 0$, so that the line passes through the origin, the sign of the denominator is usually taken the same as the sign of B. Then $\sin\alpha$ is positive and α lies between 0° and 180°.

EXAMPLE. Reduce the equation $3x + 4y - 15 = 0$ to the normal form and find the distance of the line from the origin.

SOLUTION. Here $A = 3$, $B = 4$; hence $\pm\sqrt{A^2+B^2} = \pm 5$. Dividing each term by $+5$, since the constant term is negative, expresses the equation in the normal form, namely,

$$\frac{3x}{5} + \frac{4y}{5} - 3 = 0.$$

Hence, $\cos\alpha = 3/5$; $\sin\alpha = 4/5$; and $p = 3$, the distance from the origin.

202. Distance of a Point from a Line. Let it be required to find the distance of the point $P_1(x_1, y_1)$ from the line AB when the equation of AB is given in the normal form. Draw MN through P_1 parallel to AB; and OK perpendicular to MN from the origin, O (Fig. 135). Let us denote the distance OL by p and LK by d and agree that d shall be considered positive if the origin and point P_1 are on opposite sides of the line AB, and negative if on the same side. Then the equation of MN is $x\cos\alpha + y\sin\alpha = p + d$. Since MN passes through P_1, the equation must be satisfied

by its coordinates (x_1, y_1). Hence

$$x_1 \cos \alpha + y_1 \sin \alpha = p + d,$$

and

$$d = x_1 \cos \alpha + y_1 \sin \alpha - p.$$

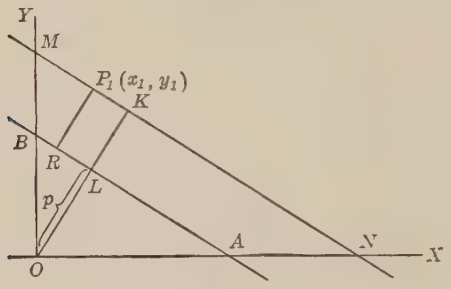


FIG. 135.

If the equation of the line is given in the form $Ax + By + C = 0$, the formula for the distance becomes

$$d = \frac{Ax_1 + By_1 + C}{\pm \sqrt{A^2 + B^2}}, \tag{20}$$

where the sign of the denominator is opposite to the sign of C. Hence the distance of a point (x_1, y_1) from a line is obtained by writing the equation of the line in the normal form and substituting x_1 for x and y_1 for y.

EXAMPLE 1. Find the distance from the point $(-2, -2)$ to the line $3x + 4y = 12$.

SOLUTION.

$$d = \frac{3(-2) + 4(-2) - 12}{+\sqrt{3^2 + 4^2}} = \frac{-26}{5} = -5.2$$

Since this distance is negative, the origin and the point $(-2, -2)$ are on the same side of the line.

EXAMPLE 2. Find the distance of the point (4, 7) from the line $3x + 4y - 6 = 0$.

SOLUTION. Dividing the equation by 5, we may write it in the normal form

$$\frac{3x}{5} + \frac{4y}{5} - \frac{6}{5} = 0.$$

Substituting 4 for x and 7 for y, we obtain

$$\text{distance} = d = \frac{3(4) + 4(7) - 6}{5} = \frac{34}{5} = 6\tfrac{4}{5}.$$

Since d is positive, the point and the origin are on opposite sides of the line.

EXERCISES

1. What is the equation of a straight line in which:

(*a*) $\alpha = 60°$, and $p = 8$.
(*b*) $\alpha = 45°$, and $p = 10$.
(*c*) $\alpha = 150°$, and $p = 5$.
(*d*) $\alpha = 135°$, and $p = 20$.
(*e*) $\alpha = 30°$, and $p = 12$.
(*f*) $\alpha = 90°$, and $p = 6$.

2. Reduce the following equations to the normal form. From the graph of the equation find the approximate value of p, and compare with the value of p obtained by reducing the equation to the normal form.

(*a*) $4x + 3y = 25$.
(*b*) $6x - 8y = 48$.
(*c*) $5x + 12y = 26$.
(*d*) $x + 2y = 10$.
(*e*) $3x - 4y = 40$.
(*f*) $3x + y = \sqrt{10}$.

3. Find the distance of the point (5, 3) from the line $4x + 3y = 10$.

4. Find the distance of the origin from the line $2x - 7y - 15 = 0$.

5. Find the area of the triangle whose vertices are (6, 0), (0, 8) and (7, 9) by calculating the length of one side and the distance of the opposite vertex from that side.

6. Find the distance between the two parallel lines $3x + 5y = 15$ and $3x + 5y = -10$. How are they located with reference to the origin?

7. Show that the point (3, 3) is on the same side of $2x + 3y = 17$ as the origin.

8. Find the equation of a line parallel to $5x + 12y = 10$, and 5 units from it.

9. Find the distance of the point (8, 11) from the line passing through (4, 0) and (5, 6).

10. Find the length of each altitude of the triangle whose vertices are (2, 1), (6, 4) and (5, −8), respectively.

11. Find the equation of a line parallel to $12x - 5y = 10$,

(a) at a distance of 2 units from it.
(b) at a distance of 6 units from it.

12. Find the distance between the parallel lines,

(a) $3x + 2y = 6$, and $3x + 2y = 20$.
(b) $8x - 6y = 20$, and $8x - 6y = 100$.

13. What system of lines is represented by the equation $x \cos \alpha + y \sin \alpha - p = 0$, if α is kept constant and p allowed to vary? If p is kept constant and α allowed to vary?

14. Find the equation of a line having its slope equal to $-4/3$ and being 7 units from the origin.

203. Bisectors of the Angles between Two Lines. Since the bisector of an angle is the locus of a point which moves

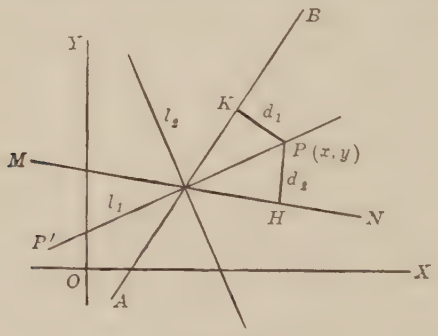


Fig. 136.

so as to be numerically equidistant from the sides, its equation is readily found. Let the equations of the sides AB and MN be

$$(AB) \qquad A_1x + B_1y + C_1 = 0,$$
$$(MN) \qquad A_2x + B_2y + C_2 = 0.$$

Let $P(x, y)$ be any point on the bisector of one of the angles between lines AB and MN. Denote the lengths of the

perpendiculars from P to the lines AB and MN by d_1 and d_2, respectively. Then

$$d_1 = \frac{A_1x + B_1y + C_1}{\pm\sqrt{A_1^2 + B_1^2}}, \quad \text{and} \quad d_2 = \frac{A_2x + B_2y + C_2}{\pm\sqrt{A_2^2 + B_2^2}}.$$

Hence the relation that must exist between x and y in order that P may be on the bisector is

$$\frac{A_1x + B_1y + C_1}{\pm\sqrt{A_1^2 + B_1^2}} = \frac{A_2x + B_2y + C_2}{\pm\sqrt{A_2^2 + B_2^2}}. \tag{21}$$

EXERCISE. Find the equations of the bisectors of the angles between the lines

(L_1) $\quad 3x - 4y - 10 = 0 \quad$ and $\quad (L_2)$ $\quad 4x + 3y - 14 = 0.$

Let $P(x, y)$ be any point on a bisector. If we denote by d_1 and d_2 the distance of this point from the lines L_1 and L_2, respectively, we have

$$d_1 = \frac{3x - 4y - 10}{5}, \quad \text{and} \quad d_2 = \frac{4x + 3y - 14}{5}.$$

Since P is on the bisector, d_1 and d_2 are numerically equal. If P is in the position shown in the figure, d_1 and d_2 are both positive, since P and the origin are on opposite sides of the line in each instance. When P is in the position P', d_1 and d_2 are both negative since P' and the origin are on the same side of the line in each instance. In any case if P is on l_1, $d_1 = d_2$. Therefore, the condition that the coordinates (x, y) must satisfy is

$$\frac{3x - 4y - 10}{5} = \frac{4x + 3y - 14}{5}$$

or

$$x + 7y = 4.$$

If P is on the bisector l_2, the condition is $d_1 = -d_2$, or

$$\frac{3x - 4y - 10}{5} = -\frac{4x + 3y - 14}{5},$$

which reduces to

$$7x - y = 24.$$

Work out
15x - 20y - 50 =

The student should be able not only to write the equations of the two bisectors, but also to tell, by aid of a figure, which equation corresponds to each bisector.

EXERCISES

1. Find the equations of the bisectors of the angles between the two lines $3x - 4y = 12$ and $4x + 3y = 10$. Show that the two bisectors are perpendicular.

2. Find the equations of the bisectors of the angles between

(*a*) $\begin{cases} 3x + 4y = 8, \\ 4x + 3y = 10. \end{cases}$ (*b*) $\begin{cases} 3x - y = 11, \\ x - 3y = 4. \end{cases}$

3. Find the equations of the bisectors of the interior angles of the triangles whose sides are:

$12x - 5y + 20 = 0$, $12x + 5y - 24 = 0$, $5x - 12y = 24$.

Show that these bisectors meet in a common point.

4. Find the locus of all points which are twice as far from $3x - 4y = 20$ as from $5x + 12y = 8$.

5. Find the locus of all points twice as far from $3x + 4y = 10$ as from $3x - 4y = 6$.

6. A triangle has sides whose equations are $x + y = 4$, $x - y = 6$, and $y = -4$. Prove that

(*a*) the triangle is isosceles.

(*b*) the bisector of an exterior angle formed by the first two sides (one side and the other extended) is parallel to the third side.

CHAPTER XIII

PROGRESSIONS AND SERIES –INTEREST FORMULAS—BINOMIAL THEOREM

204. Introduction. Arithmetic and geometric progressions are among the oldest problems of mathematics. Egyptian and Babylonian scholars had studied them as early as 2000 years before the Christian era. The papyrus of Ahmes (1650 B.C.) contains two rather complicated problems involving arithmetic progression, and one involving the summation of a geometric progression. The Babylonians used both forms of progressions in recording the phases of the moon. The Greeks were zealous students of the progressions. A decided advance in the theory was made by the Pythagoreans. In his ninth book, Euclid gave a correct formula for the summation of a geometric progression. Plato and Archimedes are said to have been well acquainted with the laws of summation. Heron made extended practical use of these laws, and Nicomachus is said to have taught the topics very fully. In 1202 A.D., Leonardo of Pisa in his *Libcr Abaci* made the summing up of these series one of the nine fundamental processes of arithmetic. Out of the association of arithmetic and geometric series term by term, grew the subject of logarithms.

Today arithmetic progressions are commonly used. The child learns to count by 1's; 2's; 5's; 10's. Time is measured at the end of equal intervals. Tables of weights and measures, simple interest on a note, tables giving values of coal, or lumber, depend on adding equal amounts. In tables of logarithms the numbers increase arithmetically.

In tables of trigonometric functions the angles increase arithmetically.

205. Arithmetic Progression. A succession of terms like

$$2, 4, 6, 8, 10, \cdots,$$

which are so related that each one after the first is obtained by adding a fixed amount to the preceding one, is called an ***arithmetic progression.*** Such a progression is indicated by the abbreviation A. P. The numbers forming the progression are called terms. If a represents the first term, d the common difference, n the number of terms considered, l the nth (or last) term, and s the sum of the terms, then a, d, n, l, and s are called the ***elements*** of the arithmetic progression.

206. Relations between the Elements. If a is the first term, then by the definition of an arithmetic progression the second term is $a + d$, the third term is $a + 2d$, and so on, so that the last term is

(1) $\quad l = a + (n - 1)d.$

The sum of an arithmetic progression may be written in either of the following ways:

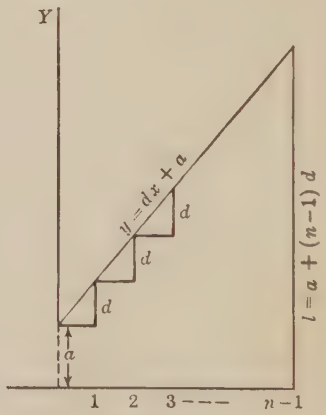


Fig. 137.

(2) $\quad s = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l.$

(3) $\quad s = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a.$

Adding (2) and (3), we have

$$2s = (a + l) + (a + l) + \cdots \text{ to } n \text{ terms.}$$

Combining terms, we have $2s = n(a + l)$, or

$$s = \frac{n(a + l)}{2}. \tag{4}$$

In the above expression $(a + l)/2$ is the average of the first and last terms. Any one of the remaining elements a, d, l, n, s may be found when three of the elements are given.

207. Arithmetic Means. The terms of an arithmetic progression between the first one and the last one are called *arithmetic means.* Thus the arithmetic means between 1 and 9 might be 3, 5, 7 since 1, 3, 5, 7, 9 form an A. P. If a single term is inserted, it is called the arithmetic mean and its value is $(a + l)/2$.

EXAMPLE. Find the last term and the sum of the terms of the arithmetic progression: 1, 4, 7, 10, etc., to 20 terms.

SOLUTION. $a = 1$, $d = 3$, and $n = 20$. Hence $l = 1 + (20 - 1)3 = 58$, the last term, and $s = (20/2)(1 + 58) = 590$, the sum.

EXERCISES

1. Determine which of the following sets of numbers form arithmetic progressions.

(*a*) 2, 4, 6, 8. (*c*) 5, 1, -3, -7. (*e*) 2, 7, 12, 17.
(*b*) 3, 1, -1, -3. (*d*) -3, 3, 9, 15. (*f*) -1, 1, 2, 3.

2. Find the last term and the sum of the following:

(*a*) $1 + 2 + 3 + 4 \cdots$ to 15 terms. (*c*) $2 + 4 + 6 \cdots$ to 10 terms.
(*b*) $1 + 2 + 3 + 4 \cdots$ to 25 terms. (*d*) $3 + 6 + 9 \cdots$ to 20 terms.

3. Compute l and s in the following:

(*a*) $a = 2$, $d = 3$, $n = 12$. (*c*) $a = 1$, $d = 2$, $n = 25$.
(*b*) $a = 3$, $d = 4$, $n = 15$. (*d*) $a = 10$, $d = 5$, $n = 18$.

4. Find the sum of the natural numbers from (*a*) 1 to 9 inclusive. (*b*) 1 to 49 inclusive. (*c*) 1 to 99 inclusive.

5. A lead ball is dropped. If it passes over 16 ft. during the first second, three times as far during the second second, five times as far during the third second, etc

(*a*) How far does it go during the fifth second? tenth second? twentieth second?

(*b*) How far does it go during the first five seconds? first 10 seconds? first 20 seconds?

6. Find a formula for d, when a, n, and l are given.

7. Find a formula for l, when a, n, and s are given.

8. Find $\sin x$ for 10′ intervals from $\sin 35°$ to $\sin 36°$. Do these values form an arithmetic progression? What is the common difference? Does the table of "$\log \sin x$" increase by a constant amount for short intervals? What is the principle underlying interpolation in the tables you have used?

9. If a clock strikes the hours, how many strokes will it make in 12 hours? In 24 hours?

10. Prove that equal multiples of the terms of an arithmetic progression form an arithmetic progression.

11. Prove that the sum of n consecutive odd integers, beginning with 1, is n^2.

12. Prove that the sum of n consecutive even integers, beginning with 2, is $n(n + 1)$.

13. Five posts are to be set equally spaced between two fixed posts 96 feet apart. How far apart must the posts be placed?

14. A carpenter wishes to purchase a pole of sufficient length for 16 rungs of a ladder. If the rungs diminish uniformly from 26 inches at the base to $14\frac{3}{4}$ inches at the top, what length will be needed?

15. A gold watch is sold by selling tickets marked from 1 to 100. The tickets are placed in envelopes and drawn from a box, the price per draw in cents being equal to the number drawn. How much money is received from the sale?

16. A debt of \$200 on a piano is to be paid in monthly installments of \$25, plus the accrued interest on the debt at 8%. How much will the first payment be one month hence? The last payment? The total amount paid?

17. On a plane inclined 30° a ball rolls approximately 8 feet in one second; 24 feet in the second second; 40 feet during the third second. How far will the ball travel in five seconds? In ten seconds?

18. A paving assessment of \$320 is payable at the rate of \$40 per year, plus the accrued simple interest at 6% on the unpaid portion. How much will be paid in all?

208. Geometric Series. A series is called a ***geometric series*** or a ***geometric progression*** if each term after the first is obtained by multiplying the preceding term by a constant number called a ratio. Such a progression is indicated by the abbreviation G. P.

Thus the series 1, 3, 9, 27, 81, etc., and 2, 8, 32, 128, etc., are geometric progressions.

If a denotes the first term, and r the common ratio, the general series may be written in the form

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}. \tag{5}$$

Since the exponent of r increases by 1 and is always 1 less than the number of the term, l, the last or nth term, will be

$$l = ar^{n-1}. \tag{6}$$

A formula for the sum of the series is easily found. Let

$$s = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \tag{7}$$

Multiplying by r, we get

$$rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{8}$$

Subtracting (8) from (7), we obtain

$$s - rs = a - ar^n.$$

Solving for s, we have

$$s = \frac{a - ar^n}{1 - r} = \frac{a(r^n - 1)}{r - 1}. \tag{9}$$

209. Graphical Representation. To represent the series

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

graphically draw the line $y = rx$. On the x-axis take $OA_1 = a$; then the ordinate $A_1P_1 = ar$, since the slope of

the line is r. Next take $P_1M_2 = A_1P_1 = ar$, and draw $P_2M_2 = ar^2$. Likewise $P_2M_3 = P_2M_2 = ar^2$, etc. Hence $OA_1 = a$, $A_1A_2 = ar$, $A_2A_3 = ar^2$, $\cdots$, $A_{n-1}A_n = ar^{n-1}$. The sum of n terms of the series is

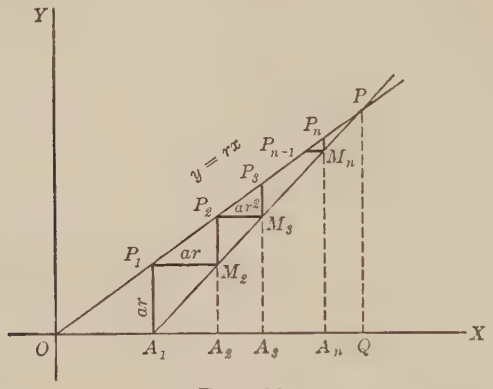


FIG. 138.

$$s = a + ar + ar^2 + \cdots + ar^{n-1} = OA_n.$$

If n is infinite, and $|r| < 1$, the limit of the sum of the infinite series is $s = OQ$, the abscissa of the point of intersection of the lines $y = rx$ and $y = x - a$. Eliminating y between these equations, x (the sum) $= a/(1 - r)$, where $|r| < 1$. See § 217.

The formula for the sum may also be obtained from the graph in another way. It is readily seen that $A_1A_n = A_nM_n$ (since the slope of $y = x - a$ is 1). But $A_1A_n = OA_n - OA_1 = s - a$. Also

$$A_nM_n = A_nP_n - M_nP_n = rs - ar^n.$$

$$s - a = rs - ar^n, \quad \text{and} \quad s(1 - r) = a - ar^n.$$

Hence

$$s = \frac{a - ar^n}{1 - r}.$$

EXAMPLE. Find the sum of the series 1, 4, 16, 64 $\cdots$ to 15 terms. Also find the last term.

SOLUTION. $a = 1$, $r = 4$, and $l = 1 \times 4^{14}$. Hence

$$s = \frac{1 - 1 \times 4^{15}}{1 - 4} = \frac{4^{15} - 1}{3}.$$

APPLICATIONS. The student will find numerous important applications for the formulas of geometric progressions in problems dealing with (*a*) business (compound interest, accumulations due to yearly payments such as insurance, installment payment of debts), (*b*) physics, (*c*) biology (Galton's law, Malthus' law), (*d*) music, (*e*) summation of series, (*f*) recreations.

EXERCISES

1. Find the last term of the series and the sum of the series

(*a*) 1, 2, 4, 8, $\cdots$ to 6 terms. To 10 terms.
(*b*) 1, 3, 9, 27, $\cdots$ to 8 terms. To 12 terms.
(*c*) 1, 1/2, 1/4, 1/8, $\cdots$ to 10 terms.
(*d*) 5, 10, 20, $\cdots$ to 8 terms.

2. The number of direct ancestors (if distinct) which an individual has is represented by the series 2, 4, 8, 16, $\cdots$. Find the total number of ancestors in the 5 generations preceding him. In the 10 generations preceding him. In the 12th generation preceding him.

3. From a grain of corn there grew a stalk which produced an ear of corn. From this ear 100 new ears were grown. If this were repeated until there were six harvests, how many bushels would there be in the sixth harvest, counting 100 ears to a bushel?

4. It is found that under certain conditions the number of bacteria in milk doubles every three hours. By how much will the number be multiplied by the end of one day?

5. From a cask containing two gallons of vinegar one quart is drawn off and the cask is then filled by pouring in water. Show that after this has been done 5 times approximately 50 per cent of the vinegar remains. SUGGESTION. $a = 2$, $r = 7/8$.

6. The four strings of a violin are tuned in fifths; that is, for two vibrations of any string there are three vibrations of the string next

higher. If the lowest or G string vibrates 196 times per second, find the number of vibrations per second made by each of the other strings.

7. A sum of $\$P$ is invested at r per cent. Show that (*a*) if the money is kept at compound interest the accumulations at the end of successive years will form a geometric progression. (*b*) If the money is kept at simple interest the accumulations will form an arithmetic progression.

8. The plunger chamber of an air pump is approximately 1/10 the total capacity. At each stroke 1/10 of the air in the receiver is removed. After 5 strokes find the amount of air remaining in the receiver.

SUGGESTION. $a = 1, r = 9/10$.

9. GALTON'S LAW OF HEREDITY. According to Galton's Law of Heredity, the parents contribute to the hereditary make-up of an individual 1/2 of all that is contributed by all ancestors; the grandparents 1/4; the great grandparents 1/8, and so on. This is shown in the following table:

GENERATION BACK.	EFFECTIVE CONTRIBUTION OF THIS GENERATION.	NUMBER OF ANCESTORS INVOLVED.	EFFECTIVE CONTRIBUTION OF EACH ANCESTOR.
1	1/2	2	1/4
2	1/4	4	1/16
3	1/8	8	1/64
4	1/16	16	1/256

Compute (*a*) the contribution of the sixth generation back; (*b*) the tenth generation; (*c*) the contribution of the six generations back; (*d*) the contribution of a single individual in the sixth generation back. Complete the table to the eighth generation.

10. The common housefly will lay the eggs, incubate and mature a new litter of approximately 200 flies every three weeks. If the flies are evenly distributed as to sex, find the approximate number of descendants of one female fly at the end of twelve weeks, supposing that all mature.

11. Assuming that each female grasshopper hatches a yearly brood of 100, evenly distributed as to sex, find the possible number of grasshoppers in the 3d generation which are descendants of a single female grasshopper. In the 5th generation. In the 10th generation.

12. MALTHUS' LAW. This law states that population increases in a geometric progression with reference to time. If the average number of descendants in each generation is four, find the total number of

descendants in the 10th generation, counting from a single pair. The series is 2, 4, 8, 16, $\cdots$. If the average length of a generation is 30 years, find the number of living descendants after 600 years, that is, the 20th generation.

13. Historical Problem. In the textbooks of the sixteenth century problems similar to the following frequently appear.

(*a*) It takes 32 nails to shoe a horse. A blacksmith agrees to drive them for the following considerations: 2 cents for the first, 4 cents for the second, 8 cents for the third, etc. Find the total cost.

(*b*) The problem is sometimes stated as follows:

1 grain of corn for the first nail, 2 grains of corn for the second, 4 for the third, etc. Counting 800 grains of corn to the ear, 100 ears to the bushel, how many bushels of corn will be required to pay for the shoeing?

210. Compound Interest. Application of the Formula $l = ar^{n-1}$. Interest is money paid for the use of money. The borrowed capital is called the principal. If interest is paid on the principal only, it is called simple interest. If interest is added to the principal periodically, forming a new principal which bears interest, it is called compound interest. If the interest is added each 12 months, 6 months or 3 months, it is said to be compounded annually, semi-annually or quarterly, respectively.

Thus if the interest is compounded annually, and if A = amount, P = principal, r = rate (expressed as a decimal) and n = number of years, the amount at the end of one year is

$$A_1 = P(1 + r); \tag{10}$$

at the end of two years is

$$A_2 = P(1 + r)^2; \tag{11}$$

at the end of three years is

$$A_3 = P(1 + r)^3, \tag{12}$$

and at the end of n years is

$$A_n = P(1 + r)^n. \tag{13}$$

If the interest is compounded semi-annually, in n years there will be $2n$ periods and the interest on one dollar for each period is $r/2$, so that the amount at the end of n years, compounded semi-annually, is

$$A_{2,\,n} = P(1 + r/2)^{2n}; \tag{14}$$

and the amount at the end of n years, compounded quarterly, is

$$A_{4,\,n} = P(1 + r/4)^{4n}. \tag{15}$$

211. Graphical Comparison of Simple and Compound Interest. In Fig. 139 the straight line AB is the graph of the equation

$$A = 1 + .06t, \tag{16}$$

where A is the amount of \$1 at 6% simple interest after t years. The curved line AC is the graph of the equation $A = 1(1.06)^t$, the amount of \$1 at 6% compound interest after t years, compounded annually.

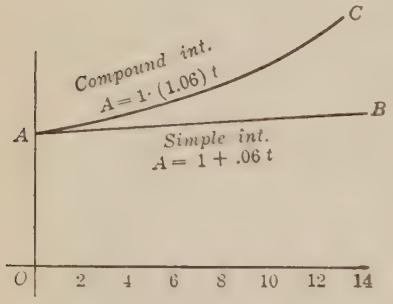


FIG. 139.

EXAMPLE. Find the amount of \$285 at 6% compound interest for 100 years.

SOLUTION. Amount $= \$285(1.06)^{100}$.

$$\log 285 = 2.45484$$
$$100 \log 1.06 = 2.53100$$
$$\log A = 4.98584$$

Hence $A = \$96.792$ (five place logarithmic table). The exact value is \$96 701.09 (eight place interest table). The student is advised to use interest tables freely.

212. Present Value. If a sum of money is payable at some future date, it is not worth as much as an equal sum payable now. The present value P of a sum of A dollars payable in n years is the principal which must be placed at compound interest at a given rate r in order to amount to A dollars in n years. The present value of \$1000 due in six years, interest at 5% compounded annually, is \$746.215, since $\$746.215(1.05)^6 = \$1000$. From the compound interest formula we have

$$A_n = P(1 + r)^n. \tag{17}$$

Solving for P, the present value, we find

$$P = \frac{A_n}{(1 + r)^n}. \tag{18}$$

It should be observed that the more remote the payment of a specified sum, the smaller its present value. The present value of \$1000 due after n years, interest compounded annually at 5% is shown in the adjacent table.

n	P
0	\$1000.
5	783.53
10	613.91
20	376.89
40	142.05

EXAMPLE. What is the present value of \$8000 due in 20 years if money is worth 5%?

SOLUTION. The present value $P = \dfrac{8000}{(1.05)^{20}}$.

$$\log 8000 = 3.90309$$
$$20 \log 1.05 = 0.42380$$
$$\log P = 3.47929$$

$P = \$3015.00$ (five-place logarithmic table).
The exact value = \$3015.11 (eight-place interest table).

EXERCISES

1. The sum of \$700 deposited with a trust company bears interest at the rate of 6%, compounded annually. Find the amount at the end of 12 years.

2. Eight years ago a man bought a lot for \$475. He estimates the increase in value to be 8% per year over the value at the end of the previous year. What is the lot worth now?

3. A building and loan association offers an opportunity for an investment to yield 8%, compounded semiannually. If \$900 is invested, to what will it amount in 4 years? 6 years? 20 years?

4. A young man borrowed \$800 from his father to pay his expenses at college. Five years later he paid his father \$1020. What rate of compound interest did he pay?

5. A certain college which has 1000 students in attendance has been increasing at the rate of 7% each year over the previous year's attendance. If the same percentage of increase should continue, how many students can they expect in ten years? In twenty years? In thirty years?

6. An acreage of timber is found to increase in value each year at the rate of 4% of the previous year's value. If its value at the beginning of the period was \$60 000, what would be the value at the end of 5 years? 10 years? 30 years?

7. Find the time required for any sum to double itself at 4%, 5%, 5.5%, 6%, 6½%, 7%, 8%.

8. The city of New York increased in population from 120 000 to 4 769 000 in the interval from 1810 to 1910. Compute the average annual rate of increase. Use the formula

$$120\,000(1 + r)^{100} = 4\,769\,000.$$

9. In 1624 the Dutch bought Manhattan Island from the Indians for \$24. Suppose that the Indians had put their money out at compound interest at 7%; how large would the accumulated amount have been in 1924?

10. The will of Benjamin Franklin, who died in 1790, directed that 1000 pounds be given in trust to the city of Boston and a similar amount to Philadelphia. The money was to be allowed to accumulate for 100 years. The amount given to Boston, approximately \$4880, accumulated to \$400 000. Find the average rate of interest earned.

11. The population of the United States increased from 7.2 millions in 1810 to 105.7 millions in 1920. Find the approximate rate of increase per year. Rate of increase per decade.

12. A young man 18 years of age is to receive from an estate the sum of $17 560 when he reaches the age of 30. Find the present value of his interest in the estate, assuming that he will reach the age of thirty, if money is worth 5%? If money is worth 6%?

13. What is the present value of $10 000 to be paid 11 years hence, money at 5%?

14. Compare the compound interest on $3000 for 10 years at 6% compounded annually with the compound interest on the same amount for the same time and rate with interest compounded semiannually.

15. A bequest of $10 000 was left to a boy 10 years of age, with instructions to pay principal and accumulated interest to him when the fund amounted to $25 000. The fund was invested at 6%, convertible semiannually. At what age should the young man receive the $25 000? Give answer to nearest half year.

213. Investment Problems. Accumulation. Application of $S = a(r^n - 1)/(r - 1)$. One of the common problems in business is to determine how large a fund will be accumulated at a specified time by making interest bearing deposits or payments at stated intervals. Or, conversely, how large must the payments be to yield a certain sum at the stated time. Such problems are common in connection with annuities, compound interest, sale of bonds for building school houses, public improvements, rent, government loans, building and loan associations, etc.

To solve such problems express the amount accumulated on each payment at the end of the time. By the compound interest formula, we may write

$$A_n = P(1 + r)^n, \tag{19}$$

or

$$A_{k,\,n} = P(1 + r/k)^{kn}. \tag{20}$$

Find the sum of these amounts, which form a geometric progression.

EXAMPLE 1. If a person makes 12 annual deposits of \$150 each, beginning now, interest 6% compounded annually, how much will there be to his credit at the end of 12 years?

SOLUTION.

The first deposit with interest will amount to $\$150(1.06)^{12}$.
The 2d deposit with interest will amount to $\$150(1.06)^{11}$.
The 12th deposit with interest will amount to $\$150(1.06)$.
Hence $A = \$150(1.06) + \$150(1.06)^2 + \cdots + \$150(1.06)^{12}$.

This is a geometric series in which $a = \$150(1.06)$, $r = 1.06$ and $n = 12$.

Substituting, $A = \dfrac{a(r^n - 1)}{r - 1} = \$150(1.06)\dfrac{(1.06)^{12} - 1}{1.06 - 1} = \$2682.32$

EXAMPLE 2. A city issued \$80 000 worth of bonds to build a school house. What sum raised annually by taxation, beginning 1 year hence, interest 5% compounded annually, will meet the face of the bonds when they mature 8 years hence?

SOLUTION.

The first sum with interest will amount to $P(1.05)^7$.
The 2d sum with interest will amount to $P(1.05)^6$.
The 8th sum with interest will amount to P.

Hence in this geometric progression $a = P$, $l = P(1.05)^7$, $r = 1.05$, $n = 8$.

Therefore $\$80\,000 = \dfrac{P[(1.05)^8 - 1]}{1.05 - 1} = \dfrac{P[(1.05)^8 - 1]}{0.05}$.

Solving for P, $P = \dfrac{80{,}000(0.05)}{(1.05)^8 - 1}$

$= \$8377$ (five-place table).

214. Life Insurance. To determine the amount of the annual premium charged for insuring one's life, the insurance company must ascertain what sum of money invested annually with interest will amount to the face of the policy at the expected time of death. (Life insurance tables show the life expectation of a person at any given age.) The annual premium includes not alone the amount necessary to pay the face of the policy, but the expenses of the company together with a reasonable profit.

From the discussion in § 213, it is evident that the amount at the end of a given period depends on whether

the payment is made at the beginning or end of the year. In making deposits in a bank periodically, the payment of premiums on life insurance, or investments in building and loan associations, the payment is made at the beginning of the term. Hence if P is the amount invested (deposited), r the rate of interest, n the number of years, then the

1st payment with interest amounts to $P(1 + r)^n$,
2d payment with interest amounts to $P(1 + r)^{n-1}$,
nth payment with interest amounts to $P(1 + r)$.

These terms form a geometric progression whose first term is $P(1 + r)$, whose last term is $P(1 + r)^n$, and whose common ratio is $(1 + r)$, where n is the number of years. Hence the amount is

$$P(1 + r)\left[\frac{(1 + r)^n - 1}{r}\right]. \tag{21}$$

If the payments are made at the end of the period, as is frequently done then the first term is P, the last term is $P(1 + r)^{n-1}$, the common ratio is $(1 + r)$, and the number of terms is n. Hence the amount, which is denoted by $S_{\overline{n}|}$, is

$$S_{\overline{n}|} = P\left[\frac{(1 + r)^n - 1}{r}\right]. \tag{22}$$

EXERCISES

1. Ten annual deposits of \$400 each are to be made, beginning now. To what sum will these deposits amount, interest at 6%, at the end of 10 years?

2. A man puts \$250 into a savings bank at the beginning of each year. If the bank pays 4% annually, what will his savings amount to at the end of 10 years?

3. A young man takes out a life insurance policy for \$5000. His annual premium is \$106.80. His life expectation is 38 years (38 pay-

ments). If these payments had been placed in a bank at 5% compound interest, to what sum would they have amounted at the end of the thirty-eighth year? Use logarithms.

4. A man saves \$600 a year, and invests in bonds yielding 6%. The interest each year is again invested in bonds yielding 6%. What will be the amount of his savings in 6 years? In 10 years?

5. A town sells \$125 000 worth of bonds to raise money to build a school house. It is planned to raise through taxation enough money to pay the yearly interest on the bonds and also set aside \$8000 per year as a sinking fund to redeem the bonds in 12 years. If the yearly amount set aside is placed at 6% compound interest, will it equal the face of the bonds in the given time?

6. If money is worth 5%, the income from what sum of money would be equal to a salary of \$3200 per year?

7. A factory has \$36 000 worth of machinery, which it estimates will last 10 years. It is now setting aside at the end of each year \$3000, interest at 5%. Will this amount, interest compounded annually, be sufficient to purchase \$36 000 worth of new machinery at the end of the time?

8. The tuition at a certain preparatory school is \$900 per year, if paid at the beginning of the year. What is the cash equivalent of four years' tuition if money is worth 6%?

9. A man buys a farm for \$24 000, paying one-half cash. He plans to pay the interest as it falls due, and to set aside at the end of each year a fixed sum which at 5% compound interest will enable him to discharge the debt in 10 years. How much will he need to set aside annually?

10. Each year a merchant deposits \$900 in the bank and allows it to accumulate at 6% compound interest. How much will he have to his credit just after he has made the 10th deposit?

11. A city is to issue 12-year bonds to the amount of \$120 000 for the erection of public schools, and it is desired to establish a *sinking fund* to provide for the extinction of the debt when due. How much must be deposited in the sinking fund at the end of each year, money being worth 5% compounded annually?

215. Valuation of a Bond. When a government or corporation issues a bond, it promises to pay on a given date

the sum F dollars specified in the bond and in the meantime to pay at stated intervals a certain amount of interest.

Since the interest rate prevailing in the money market is seldom the same as the rate named in the bond, the market value of the bond is usually different from its face value. If an investor desires a specified investment rate, the price P he is willing to pay on purchasing the bond is

$$P = \text{present value of } F \text{ due on redemption date} + \text{present value of interest payments},$$

where the present values are computed under the investor's rate.

EXAMPLE. A \$1000 city bond, maturing in 8 years, bears interest at $4\frac{1}{2}\%$, payable annually. Find its present value, assuming money to be worth 6% compounded annually.

SOLUTION. The city pays \$45 interest each year, beginning one year hence, until maturity; the present value of the interest payments, money at 6%, is

$$\frac{45}{1.06} + \frac{45}{(1.06)^2} + \cdots + \frac{45}{(1.06)^8}.$$

The sum of these terms, which form a geometric progression, is

$$S = \frac{\$45}{(1.06)^8} \times \left[\frac{(1.06)^8 - 1}{.06}\right] = \$279.44$$

The present value of the principal, \$1000, payable 8 years hence, is

$$S_1 = \frac{1000}{(1.06)^8} = \$627.41$$

The total $S + S_1 = \$906.85$, the present value of the bond.

EXERCISES

1. A man buys a farm. The payments are \$1000 cash and \$8000 in 5 years without interest. Find the present cash value if money is worth 6%.

2. A note for \$1500 maturing in 5 years bears interest at 4%. Find its present value if money is worth 6%.

3. A note for \$2000 maturing in 10 years bears $7\frac{1}{2}\%$ interest. Find its present value if money is worth 4%.

4. A house is sold on the following terms. The purchaser is to pay \$3000 cash and \$1000 at the end of each year for five years, without interest. Find the cash equivalent if money is worth 5%.

5. As security for a loan, a man deposits a 10-year endowment life insurance policy for \$5000. Find the present value of the security, assuming that all payments of premium will be met, interest being computed at 5%.

6. A \$10 000 $4\frac{1}{2}\%$ bond is redeemable in 5 years. Find the present purchase price that will net 6% interest.

216. Geometric Means. The first and last terms of a geometric progression are called the extremes, and the intervening terms the geometric means. In a geometric progression of three terms, the middle term is called the geometric mean. By the aid of the formula for the last term any number of geometric means can be inserted between two given numbers.

EXAMPLE. Insert 3 geometric means between 2 and 162. Here $a = 2$, $l = 162$, $n = 5$. Substituting these values in $l = ar^{n-1}$, we find $2 \times r^4 = 162$, whence $r^4 = 81$, and $r = \pm 3$. Hence the required geometric means are 6, 18, 54, or -6, 18, -54.

217. Infinite Geometric Series. If the number of terms in the series $a + ar + ar^2 + \cdots$ is unlimited, it is called an infinite geometric series. If the sequence of the partial sums a, $a + r$, $a + ar + ar^2$, etc., approaches a definite number s as a limit, s is called the sum of the infinite series

$$a + ar + ar^2 + ar^3 + \cdots. \tag{23}$$

Consider the geometric series

$$1 + 1/2 + 1/4 + 1/8 + \cdots,$$

for which $a = 1$, and $r = 1/2$. The partial sums may be

represented graphically as follows:

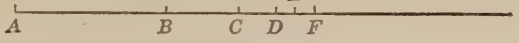


FIG. 140.

$$AB = 1 = BF, \qquad AC = 1 + 1/2,$$
$$AD = 1 + 1/2 + 1/4, \qquad AE = 1 + 1/2 + 1/4 + 1/8.$$

As the number of terms increases, the sum increases, approaching 2 in such a way that it may be made to differ from 2 by less than any assigned positive number, however small. Hence 2 is said to be the sum of the infinite series

$$1 + 1/2 + 1/4 + 1/8 + \cdots,$$

and this series is said to be *convergent*.

The formula for the sum of n terms of a geometric progression is

$$s = \frac{a - ar^n}{1 - r}, \tag{24}$$

which may be written in the form

$$s = \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \tag{25}$$

If r is less than 1 in numerical value, the successive terms continually decrease in numerical value. By increasing n, the value of r^n may be made as small as we please. Hence as n increases beyond all bounds, the limit of ar^n is zero, and

$$s = \frac{a}{1 - r}. \tag{26}$$

EXERCISES

1. Insert one geometric mean between 3 and 48.

2. Insert two geometric means between 2 and 686.

3. Find the geometric mean between 4 and 6.

4. Find the sum of the following infinite geometric progressions.

(*a*) $1 + 1/4 + 1/16 + 1/64 + \cdots$.
(*b*) $1 + 1/3 + 1/9 + 1/27 + \cdots$.
(*c*) $1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots$.
(*d*) $1/2 + 1/4 + 1/8 + 1/16 + \cdots$.

5. What common fraction gives rise to the repeating decimal $.444\cdots$.

HINT. Find the sum of the series $0.4 + 0.04 + 0.004 + \cdots$.

6. An elastic ball falls 10 feet and at each fall rebounds 2/5 of the distance it fell. Find the distance the ball has traveled before coming to rest.

7. To divide an octave into twelve equal semitones is mathematically the same as inserting eleven geometric means between 1 and 2, that is, 1 and 2 are the first and last terms of a G. P. of 13 terms. Find the number of vibrations in each semitone from middle C (256 vibrations) to the octave above, 512 vibrations.

8. The altitude from the vertex of a right angle on the hypotenuse of a right triangle is a geometric mean between the segments of the hypotenuse. Find the length of this perpendicular if the segments into which it divides the hypotenuse are 6.4 ft. and 12. Find the area of the triangle.

218. Expansion of a Binomial. By successive multiplication we obtain the following identities.

$$(a + b)^2 = a^2 + 2ab + b^2.$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

If the expanded form of $(a + b)^3$ is multiplied by $(a + b)$,

using only the coefficients, we have

$$\begin{array}{r} 1+3+3+1 \\ \underline{1+1} \\ 1+3+3+1 \\ \underline{1+3+3+1} \\ 1+4+6+4+1 \end{array}$$

from which the law of formation of the coefficients 1, 4, 6, etc., is evident. Writing down the coefficients of the powers of $(a+b)$ in order, we have the arithmetic triangle shown below.

TABLE OF BINOMIAL COEFFICIENTS: $(a+b)^n$

						1						$n=0$	1										
					1		1					$n=1$	1		1								
				1		2		1				$n=2$	1		2		1						
			1		3		3		1			$n=3$	1		3		3		1				
		1		4		6		4		1		$n=4$	1		4		6		4		1		
	1		5		10		10		5		1	$n=5$	1		5		10		10		5		1
1		6		15		20		15		6		1	$n=6$	1		6		15		20		15		6		1

PASCAL'S TRIANGLE. POWER. ARITHMETIC TRIANGLE.

In the arithmetic triangle, after the first line, each number is the sum of the number above it and the number to the left of the latter. Thus 15 in the seventh line equals 10+5, etc.

219. Historical Note. The binomial theorem for cases in which $n = 2$ or $n = 3$ was known to Euclid 300 B.C., as well as to other Greek mathematicians of about the same period. The Persian mathematician Omar Khayyam (died 1123 A.D.) extended the rule to other positive integers. It is said that a work appeared in China in 1303, which contained the binomial coefficients arranged in triangular

form, but the truth of this is disputed. The so-called Pascal triangle of coefficients was printed in 1665.

Sir Isaac Newton while an undergraduate at Cambridge proved that the coefficients of any power could be found without knowing the coefficients of the preceding power, thus showing that the coefficients of any power n of a binomial are functions of the exponent n. The significance of the theorem when n is fractional was discovered by Sir Isaac Newton and made known in a letter dated 1676. The proof for the general case when n is any real or complex number was finally effected by the brilliant Norwegian mathematician Abel in 1826.

220. Binomial Theorem. The expansion of $(a + b)^n$ in § 218 forms a series and may be written as follows.

$$(a + b)^2 = a^2 + 2ab + b^2.$$

$$(a + b)^3 = a^3 + 3a^2b + \frac{3 \cdot 2}{1 \cdot 2} ab^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} b^3.$$

$$(a + b)^4 = a^4 + 4a^3b + \frac{4 \cdot 3}{1 \cdot 2} a^2b^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} ab^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} b^4.$$

It will be seen that each term is derived from the preceding term by a definite law, which may be stated as follows.

If n represents the exponent of $(a + b)$, we may make the following statements.

(1) Law of Exponents:

(*a*) *The exponent of a in the first term is n and decreases by* 1 *with each successive term.*

(*b*) *The exponent of b in the second term is* 1 *and increases by* 1 *in each succeeding term, becoming n in the last term if n is a positive integer.*

(2) Law of Coefficients:

(*a*) *The coefficient of the first term is* 1, *that of the second term* n, *that of the third term* $\frac{n(n-1)}{1\cdot 2}$, *etc. The coefficient of any term can be found from the next preceding term by multiplying the coefficient of that term by the exponent of* a *and dividing by one more than the exponent of* b.

(*b*) *If the exponent* n *is a positive integer, the number of terms is* $n + 1$. *If* n *is not a positive integer, the series has an infinite number of terms.*

These laws will be assumed here without proof.

221. Approximate Roots. The binomial theorem furnishes a convenient method of approximating roots.

(*a*) Square Root.

Let

$$\sqrt{x^2 + a} = (x^2 + a)^{1/2} = (x^2)^{1/2} + \frac{1}{2}(x^2)^{-1/2}a - \frac{1}{8}(x^2)^{-3/2}a^2 \cdots = x + \frac{a}{2x} - \frac{a^2}{8x^3} + \cdots.$$

If a is small in comparison with x, a first approximation is given by $x + a/(2x)$, since $\sqrt{x^2 + a} = x + a/(2x)$, nearly.

Example 1. Find the square root of 27.

$\sqrt{27} = (25 + 2)^{1/2} = 5 + 2/10 = 5.2$, approximately.

Example 2.

$\sqrt{78} = (81 - 3)^{1/2} = 9 - 3/18 = 8.83$, approximately.

(*b*) Cube Root.

Let

$$\sqrt[3]{x^3 + a} = (x^3 + a)^{1/3} = (x^3)^{1/3} + \frac{1}{3}(x^3)^{-2/3}a - \frac{1}{9}(x^3)^{-5/3}a^2 \cdots = x + \frac{a}{3x^2} - \frac{a^2}{9x^5} + \cdots.$$

If a is small in comparison with x, a first approximation is given by $x + a/(3x^2)$, since $\sqrt{x^3 + a} = x + a/(3x^2)$, nearly.

EXAMPLE. Find the cube root of 130.

$$\sqrt[3]{130} = \sqrt[3]{125 + 5} = 5 + \frac{5}{75} = 5.06 +.$$

EXERCISES

1. Expand each of the following expressions by the binomial theorem.

(*a*) $(x + y)^3$.	(*f*) $(a + b)^6$.	(*k*) $(a + 3)^3$.	(*p*) $(x - 2)^3$.
(*b*) $(a + 2b)^3$.	(*g*) $(x + 3y)^3$.	(*l*) $(a + 5)^3$.	(*q*) $(a - 5)^3$.
(*c*) $(a + b)^4$.	(*h*) $(x + 5y)^3$.	(*m*) $(a + 1)^4$.	(*r*) $(a - 10)^3$.
(*d*) $(a + b)^5$.	(*i*) $(a + 1)^3$.	(*n*) $(a + 1)^6$.	(*s*) $(x + 4)^3$.
(*e*) $(a + b)^7$.	(*j*) $(a + 2)^4$.	(*o*) $(x - y)^3$.	(*t*) $(x - 6)^3$.

2. Find 7 terms of the expansion of $(a + b)^n$, of $(1 + 1/n)^n$, of $(1 + x)^n$.

3. Find three terms of the expansion of each of the following expressions.

(*a*) $(3a + b)^{10}$.	(*d*) $(1 + 1/2)^4$.	(*g*) $(1 + .5)^4$.	(*j*) $(30 + 5)^2$.
(*b*) $(x + 10y)^8$.	(*e*) $(1 - 1/2)^4$.	(*h*) $(1 + .2)^5$.	(*k*) $(3 + 1/2)^2$.
(*c*) $(x + y/10)^6$.	(*f*) $(1 + 1/3)^3$.	(*i*) $(20 + 5)^2$.	(*l*) $(4 - 1/2)^2$.

4. Expand $(1 + 1/2)^4$ and $(2 - 1/2)^4$ and compare results.

5. Find the approximate square root of 26 by expanding the binomial $(25 + 1)^{1/2}$.

6. Find the approximate cube root of 28 by expanding the binomial $(27 + 1)^{1/3}$.

7. Find the approximate square roots of 10, 17, 37, 51, 66, 83, 102, 8, 15, 24, 35, 47, 61, 80, 97, 140.

8. Find the approximate cube roots of 9, 29, 65, 126, 220, 10, 26, 62, 123, 212.

9. Using the binomial theorem, compute, correct to two decimal places, the amount at the end of 10 years of \$200, interest at 5%, compounded annually.

$$A = \$200(1 + .05)^{10}.$$

10. Construct a graph, in which the binomial coefficients in Pascal's triangle are used as ordinates and the numbers of the corresponding

terms as the abscissas. Construct the graph for $n = 3$; for $n = 5$; for $n = 7$.

11. Prove that the sum of the binomial coefficients of order n when n is a positive integer is 2^n.

SUGGESTION. $(x + y)^n = x^n + nx^{n-1}y + \cdots + y^n$.

Putting $x = 1$, and $y = 1$, we have $(1 + 1)^n = 2^n$ = sum of binomial coefficients. Verify the result from the table for particular values of n.

12. The sum of the odd numbered coefficients in the binomial expansion when n is a positive integer is equal to the sum of the even numbered ones and each is equal to 2^{n-1}. Verify this statement from the table for particular values of n.

SUGGESTION. Let $x = 1$, and $y = -1$ in the binomial formula.

13. Prove that

$$(1 + 1/n)^n = 1 + 1 + \frac{1 - 1/n}{2!} + \frac{(1 - 1/n)(1 - 2/n)}{3!} + \cdots,$$

where $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3$, $4! = 1 \cdot 2 \cdot 3 \cdot 4$, etc.

CHAPTER XIV

LAWS OF GROWTH—EXPONENTIAL FUNCTION

222. Compound Interest Function. The function

$$A = P(1 + r)^n,$$

besides being of fundamental importance in finance, is one of the important laws of nature and deserves special consideration. Such an expression as x^2 in which a variable is raised to a fixed power is called a power function. An expression such as 2^x in which a constant is raised to a variable power is called an exponential function. These functions increase in very different ways, as is shown by the following table.

x	0	1	2	3	4	6	8	10
2^x	1	2	4	8	16	64	256	1024
x^2	0	1	4	9	16	36	64	100

Definition. If any quantity y varies with another, x, in such a way that the rate of change in y is constantly proportional to the value of y, it is said to vary according to the ***compound interest law.*** Any such quantity may be expressed by the formula

$$y = Pe^{rx}, \tag{1}$$

where P is the value of y when $x = 0$, e is a special constant explained in § 223, and r is a fixed rate. This law is frequently expressed by saying that two quantities are so related that one quantity, y, varies in geometric progression as the other, x, varies in arithmetic progression.

That is, if x is made to take on a series of values differing by a constant, as 2, 4, 6, etc., the differences between the logarithms of the corresponding values of y are constant.

223. Exponential Series. Consider the expression

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}. \tag{2}$$

By the binomial theorem,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\left(\frac{1}{n}\right) + \frac{nx(nx-1)}{1\cdot 2}\left(\frac{1}{n^2}\right) + \cdots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \cdots.$$

Moreover

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left[1 + 1 + \frac{1-\frac{1}{n}}{2!} + \cdots\right]^x. \tag{3}$$

It can be shown that these expansions converge for all values of n numerically greater than 1, and that the right-hand sides of these two equations approach equal limits as n becomes infinite. Assuming the truth of these statements and remembering that the limit of any constant divided by n as n increases without limit is zero, we may write

$$\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{4}$$

A complete proof is given in more advanced books. The sum of the infinite series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

is called the *natural base of logarithms*, and is represented by the letter e. Hence the preceding formula becomes

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{5}$$

224. To Calculate the Value of e. The value of e is given by the series

$$e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots, \tag{6}$$

where $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3$, $4! = 1 \cdot 2 \cdot 3 \cdot 4$, etc. The computation may be arranged as follows:

$$\begin{aligned} 1 &= 1.00000, \\ 1 &= 1.00000, \\ \frac{1}{2!} &= 0.50000, \\ \frac{1}{3!} &= 0.16667, \\ &\cdots\cdots\cdots \end{aligned}$$

Carrying this to the term $1/7!$, and adding, we find $e = 2.71828$, which is correct to five decimal places. Like the number 3.1416, called π, the number e occurs frequently in problems. As stated in § 223, it is called *the natural*

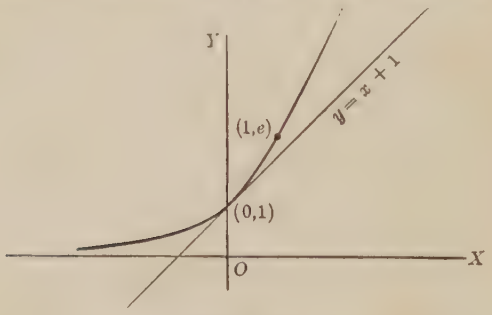


Fig. 141.

base of logarithms, or *the Napierian base of logarithms*, in honor of John Napier (1550–1617), the inventor of logarithms. In 1873 it was proved that e could not be the root

of any algebraic equation with integers as coefficients. In 1882 the same fact was proved concerning π.

225. Graph of Standard Exponential Curve. The curve $y = e^x$ will hereafter be referred to as the *standard exponential curve.* Its graph is shown in Fig. 141. Note the following properties of $y = e^x$, and compare with $y = 2^x$, $y = 3^x$ and $y = 10^x$.

1. It crosses the y-axis at $+ 1$, with a slope of 1.
2. It goes through the point $(1, e)$.
3. The negative end of the x-axis is an *asymptote,* that is, the curve will come as close as we please to the negative end of the x-axis, but will not cross it.

Some exponential curves closely related to $y = e^x$ are $y = ke^x$, and $y = ke^{ax}$.

226. The Curve $y = e^{-x}$. The graph of $y = e^x$, if turned through 180°, using OY as the axis of turning, takes the position shown in Fig. 142. This is equivalent to changing

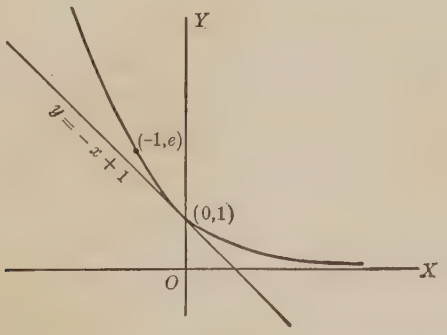


Fig. 142.

x into $- x$; hence the equation becomes $y = e^{-x}$.

1. It crosses the y-axis at $+ 1$, with a slope of $- 1$.
2. It goes through the point $(- 1, e)$.
3. The positive end of the x-axis is an asymptote.

NOTE. The relation $y = ke^{ax}$, called *the snowball law, the law of organic growth*, or, by Lord Kelvin, *the compound interest law*, is important. It is commonly found in problems of finance (compound interest), growth of timber, increase in bacteria, etc. The related curve $y = ke^{-ax}$ called the law of "organic decay" is commonly used to determine the altitude, the atmospheric pressure being given. It also occurs in problems dealing with the healing of the surface area of a wound, decomposition of radium, transmission of light through media, loss of heat or electrical energy, and the damped vibrations of a pendulum or flywheel.

227. Damped Oscillations. The *die-away* phenomena of an electric current after the source has been cut off are shown by the graph of $i = ke^{-at}$. If the current is an alternating current, the intensity at any instant is given by some such equation as $i = e^{-10t} \sin 256t$. The damping factor e^{-10t} rapidly decreases, giving the waves a continually smaller amplitude.

For convenience in plotting, values of e^x and of e^{-x} are given in Table Va, p. 392. To find the value of $e^{3.4}$ look in the column with the caption x for 3 and then pass to the right under the caption .4 The desired value is in the column under e^x, and is 30. That is, $e^{3.4} = 30$. The next value to the right of this under e^{-x} is .03 Hence $e^{-3.4} = .03$

EXERCISES

1. Plot the graphs of each of the following and compare with the graph of $y = e^x$.

(*a*) $y = e^{2x}$. (*b*) $y = e^{-2x}$. (*c*) $y = 2^x$.

2. By means of the table find P in each of the following equations.

(*a*) $P = e^{2x}$, if $x = 1.3$
(*b*) $P = e^{3x}$, if $x = 1.4$
(*c*) $P = e^x$, if $x = 5.9$
(*d*) $P = e^x$, if $x = 3.2$
(*e*) $P = e^{4.4}$.
(*f*) $P = e^{2.2}$.

3. By means of the table find P in each of the following equations.

(*a*) $P = e^{-2x}$, if $x = 1.4$
(*b*) $P = e^{-x}$, if $x = 5.4$
(*c*) $P = e^{-.1}$.
(*d*) $P = e^{-3}$.
(*e*) $P = e^{-x}$, if $x = 5$.
(*f*) $P = e^{-2}$.

4. Find the approximate value of x in each of the following equations.

(*a*) $60 = e^x$. (*b*) $314 = e^x$. (*c*) $365 = e^x$. (*d*) $18 = e^x$.

5. Halley's law states that if P is the pressure, as measured by the height of the mercury column in inches, and h is the elevation expressed in feet, then

$$P = 29.92e^{-h/26\,200}.$$

Find the pressure in inches of mercury at the following altitudes:

(*a*) sea level, $h = 0$.
(*b*) a hill, elevation 2620 ft.
(*c*) an elevation of 1 mi.
(*d*) a mountain, elevation 2 mi.
(*e*) a mountain, elevation 4 mi.
(*f*) Mt. Everest, 29 000 ft.
(*g*) an airplane, 42 000 ft.
(*h*) an elevation of 10 mi.

6. Show that if in Halley's law h is measured in miles, then the pressure in inches of mercury is given approximately by

$$P = 29.92e^{-h/5}.$$

7. Mt. Wilson, near Pasadena, Calif., has an elevation of 5900 feet. Find (*a*) the approximate height of the barometer, and (*b*) the atmospheric pressure in pounds per sq. in.

SUGGESTION. The pressure in pounds per sq. in. is $P/(2.042)$.

8. In a record ascent the barometer of an airplane registered 7.48 inches of mercury. Find the altitude reached.

9. Assuming the law in Ex. 6 to hold, at what height will the pressure be half as great as at sea level?

10. It is sometimes stated that the atmosphere extends to a height of 500 miles. If the pressure in inches of mercury at sea level is 30 in., compute the pressure at a height of 3 mi., 4 mi., 5 mi., 10 mi.

11. State the elevation of mountains or table-lands with which you are acquainted, and compute the pressure as measured by the height of the column of mercury.

12. If fluid friction is used to retard the motion of a flywheel making V_0 revolutions per minute, the number of revolutions after t minutes is given by $V = V_0e^{-kt}$. If $k = 1.5$, how long must the friction be applied to reduce the number of revolutions from 160 to 48 per minute?

13. At an elevation of h miles the pressure P of the atmosphere per square inch is $P = 14.7e^{-kh}$. If at a height of 4 miles $P = 6.6$, find the value of k.

14. An electric current does not vanish instantly when the E.M.F. is cut off, but decreases at a constant percentage rate, according to the formula $i = ae^{-kt}$. If $a = 20$ amperes and $k = 100$, find i after .05 second.

228. Growth in Nature. In nature quantities grow (increase) in a manner similar to the way a sum of money would grow (increase) at compound interest, if the interest were compounded exceedingly often or continuously. We have already seen that the amount of P dollars at $r\%$ compound interest for n years is

$$A_n = P(1 + r)^n. \tag{7}$$

If the interest is compunded k times a year, then

$$A_{k,\,n} = P\left(1 + \frac{r}{k}\right)^{kn}. \tag{8}$$

Now, if k increases indefinitely, in order to simplify the expression, we let $r/k = 1/z$. Whence $k = rz$, and $A_{rz,\,n} = P(1 + 1/z)^{znr}$. If k, and hence z, increases indefinitely, then the limit of $(1 + 1/z)^z = e = 2.71828$ Hence, $A_{\infty,\,n} = Pe^{nr}$, which is the amount of any principal P after n years with interest compounded *continuously* at the annual rate of $r\%$.

Similarly any physical quantity P which grows continuously at a constant rate $r\%$ per unit of time has after t units the value Q, where

$$Q = Pe^{rt}. \tag{9}$$

Example. The number of bacteria in a culture increased continuously at a rate (per hr.) which was always 12% of the number then present. If the original number was 1000, how many were there after 10 hours? 25 hours?

$N = 1000e^{.12t}$. If $t = 10$ hours, then $N = 1000e^{1.2}$.

From the table,

$$e^{1.2} = 3.32$$

Hence

$$N = 3320.$$

229. Depreciation. In a very large number of instances there is a continuous decrease instead of an increase. The equation then appears in the form

$$Q = Pe^{-nr}. \tag{10}$$

EXAMPLE. A building is assumed to decrease continuously in value at the rate of 8% per year. If P is its present value, its value n years from now is Q, where $Q = Pe^{-.08n}$. After 9 years its value is $Q = Pe^{-.72}$.

NOTE. Other applications for the above formula are found in decrease in radium, progress of the healing of a wound, etc.

EXERCISES

1. Find the amount of \$1000 at the end of 12 years, interest at 6% compounded continuously. After 20 years.

$$A = 1000e^{0.06\times 12} = 1000e^{.72}.$$

2. In 1870, Portland, Oregon, had a population of 8300. Its growth was similar to that of an investment with interest at 8% compounded continuously. Estimate the population for 1890. For 1910. For 1930.

3. The growth in a tree was such that its diameter increased continuously at the annual rate of 10% for 8 years. If the original diameter was 6 inches, find the diameter at the end of the period.

4. An office building worth \$100 000 decreased in value at the annual rate of 5% for 12 years. Find its value at the end of the period, if the value decreased continuously.

5. A gram of radium is dissipated continuously at the rate of 40% per 1000 years. Find the amount left after 5000 years.

6. The number of bacteria in a culture increases at a rate per hour always equal to 30% of the number. Find the number of bacteria at the end of 10 hours if the original number was 1000. At the end of 18 hours.

7. A nebula whose present temperature is 16 000° is losing heat continuously at the rate of 25% each million years. Find its temperature after 10 million years. After 20 million years.

8. Sunlight transmitted down into deep water loses in intensity continuously at the rate of 50% for each 90 feet of depth. Find the depth at which its intensity will be 10% of its original value.

9. An avalanche is started by a large snowball weighing 10 lbs. As it descends it increases in weight continuously at the rate of 100% every 528 feet. What will be its weight after it has traveled for .5 of a mile?

10. The so-called "hay bacteria" have been found to increase at a rate of 100% of their number every 20 minutes. Starting with 100, how many bacteria would there be in one hour, considering the increase to be continuous? In 1 hour 40 minutes?

11. Cholera bacteria have been found to double their number, under favorable conditions, every 30 minutes. Starting with 1000, how many would there be in 2 hours if they increase continuously?

12. Radium decomposes at a rate (per century) which at any instant is equal to 4.1% of the quantity (Q) remaining. If the present amount is 50 mg., find the amount remaining after 1000 years. 10 000 years. The decrease is continuous.

13. Find the amount of \$1000, interest at 6% per year for 10 years compounded continuously. Compare the result with the amount of \$1000 at 6% for 10 years compounded annually.

14. In a certain culture 10 000 bacteria were present at the time $t = 0$, and 80 000 at the time $t = 8$. Find the rate of increase, the increase being continuous.

15. A fireless cooker oven is 300° above the temperature of the room when $t = 0$, and 240° one hour later. How long will it be before it will be 40° above the room temperature if the decrease is continuous?

16. The speed V of a rotating wheel after the power was cut off decreased at a rate per second which at every instant was 20% of V itself. If V was originally 800, what was the value of V after 5 seconds? After 10 seconds? When was $V = 80$?

17. If the sun has at present a temperature of 12 000°, and the temperature is decreasing continuously at the rate of 20% each million years, find the temperature after 5 million years.

CHAPTER XV

CONIC SECTIONS

230. Applications of Algebra to Geometry. The study of the cone, begun by the Greek mathematician Eudoxus, was continued by such investigators as Euclid (about 300 B.C.), and Archimedes (about 225 B.C.). Archimedes computed the area of a parabolic segment and the area of an ellipse. Euclid's book on conic sections was superseded by a treatise written by the "Great Geometer," Apollonius of Perga (about 225 B.C.), who discovered that the parabola, ellipse, and hyperbola could be cut from one and the same cone. Many centuries later the practical importance of conic sections began to be emphasized. Kepler (1571–1630 A.D.) discovered that the path of the earth about the sun is an ellipse. Galileo found the path of a projectile to be a parabola if air resistance is neglected. More recently it has been found that large groups of scientific problems in astronomy, geology, physics, mechanics, and architecture depend upon a knowledge of these curves for their solution. By means of conic sections, modern astronomy has explained the motions of the heavenly bodies and has freed mankind from much of the superstition and abject terror formerly produced by the appearance of a comet or other unusual celestial phenomena.

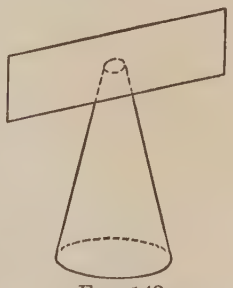


FIG. 143.

Although commonly known as branches of mathematics, geometry and algebra remained unrelated subjects until 1637 when Descartes published his method for the treatment

of geometry by means of coordinates and equations. His method, systematic and powerful, is the basis of the work in this chapter.

231. The General Equation of the Circle. In § 19 we found that the equation of the circle whose center is at (a, b) and whose radius is r is:

$$(x - a)^2 + (y - b)^2 = r^2, \tag{1}$$

or

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0. \tag{2}$$

This equation is satisfied by the coordinates of any point on the circle. Conversely, if its coordinates x and y satisfy this equation, the point P is at the distance r from the center and consequently lies on the circle.

Equation (2) may be written in the form

$$x^2 + y^2 + Dx + Ey + F = 0, \tag{3}$$

where

$$D = -2a, \qquad E = -2b, \qquad F = a^2 + b^2 - r^2. \tag{4}$$

Equation (3) is called the general form of the equation of the circle. The general equation of the second degree has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Comparison with (3) shows that this equation can be reduced to the form (3) if $B = 0$ and $A = C$ by dividing by A. Hence we have the following theorem.

Any equation of the second degree in the variables x and y represents a circle when, and only when, the coefficients of x^2 and y^2 are equal and the term in xy is wanting.

232. Length of a Tangent. Let l be the length of the tangent P_1T; (a, b) the coordinates of the center C, and r the radius of the circle. Then in the right triangle P_1TC,

$$l^2 = \overline{P_1C}^2 - \overline{CT}^2. \tag{5}$$

From the distance formula $\overline{P_1C}^2 = (x_1 - a)^2 + (y_1 - b)^2$; also $CT = r$. Substituting these values in (5),

$$l^2 = (x_1 - a)^2 + (y_1 - b)^2 - r^2. \tag{6}$$

Whence

$$l = \sqrt{(x_1 - a)^2 + (y_1 - b)^2 - r^2}. \tag{7}$$

It should be observed that the quantity under the radical in the expression for l is in the same form as the left hand side of equation (1) after r^2 is transposed to the left, the coordinates of P_1 being substituted for the variables x and y. Hence, to find the length of a tangent to a circle from a given point, transpose all terms of the given equation to the first member, substitute in the expression thus formed the coordinates of the given point, divide by the coefficient of x^2, and extract the square root.

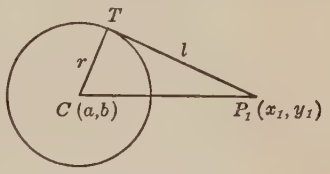


FIG. 144.

EXAMPLE. Find the length of the tangent drawn from (5, 6) to the circle $x^2 + y^2 - 4x - 2y + 1 = 0$.

Using (7), we have,

$$l = \sqrt{5^2 + 6^2 - 20 - 12 + 1} = \sqrt{30}.$$

EXERCISES

1. What can be said of the coefficients D, E and F in the general equation of the circle if the circle has its center:

(*a*) On the x-axis? On the y-axis?
(*b*) Touches both axes?
(*c*) Passes through the origin?
(*d*) Has its center on the line $x - y = 0$?
(*e*) Is tangent to both axes and has its radius equal to 5?

2. Find the length of the tangent from the point

(*a*) (5, 6) to the circle $x^2 + y^2 - 12 = 0$.
(*b*) (− 4, 3) to the circle $x^2 + y^2 - 4x - 5 = 0$.
(*c*) (− 3, 7) to the circle $x^2 + y^2 - 6x - 8y = 0$.
(*d*) (3, 7) to the circle $x^2 + y^2 = 16$.

3. Find the length of the tangent to the circle $x^2 + y^2 = 25$ from the point (*a*) (11, 2); (*b*) (4, 3); (*c*) (2, 3). What does the zero value in the answer to (*b*) mean? The imaginary answer in (*c*)?

4. The diagram of a bridge shows an approach of uniform grade run from the point (2, 10) so as to be tangent to a circular arch whose equation is $(x - 60)^2 + y^2 - 400 = 0$. Find (*a*) the length of the approach, (*b*) the slope of the approach.

5. The length of the tangent from the point (x, y) to the circle $x^2 + y^2 - 4x = 0$ is equal to the length of the tangent from the same point to the circle $x^2 + y^2 - 6y = 0$. Show that the locus of (x, y) is a straight line through the origin.

6. The length of the tangent from the point (x, y) to the circle $x^2 + y^2 - 6x - 10 = 0$ is twice the length of the tangent from the same point to the circle $x^2 + y^2 - 6y + 2 = 0$. Show that the point lies on the circle $x^2 + y^2 + 2x - 8y + 6 = 0$.

7. A power plant has two wheels 1 ft. and 6 ft. in radius, respectively. They are connected by a belt. If the centers of the wheels are 12 feet apart, find the length of that part of the belt which is the common tangent to the two wheels.

8. Prove that each of the following loci is a circle, and find the radius and coordinates of the center for each of them.

(*a*) A point moves so that the sum of the squares of its distances from (− 2, 0) and (4, 0) is always 26.
(*b*) A point moves so that its distance from (2, 4) is twice its distance from (− 3, 0).
(*c*) A point moves so that the square of its distance from (3, 5) is equal to its distance from $3x + 4y - 10 = 0$.

9. Prove that each of the following loci is a circle.

(*a*) A point moves so that the sum of the squares of its distances from two fixed points is constant.

SUGGESTION. Choose the fixed points as $(- a, 0)$ and $(a, 0)$.

(*b*) A point moves so that the square of its distance from a fixed point is proportional to its distance from a fixed line.

10. A point moves so that its distances from two fixed points are in a constant ratio k. Show that this locus is a circle except when $k = 1$. If $k = 1$, the locus is a straight line.

233. Tangent to the Circle, Slope Given. Let

$$x^2 + y^2 = r^2 \tag{8}$$

be the equation of a circle whose center is at the origin, and let m be the given slope of the tangent. Then the equation of the tangent is in the form

$$y = mx + b. \tag{9}$$

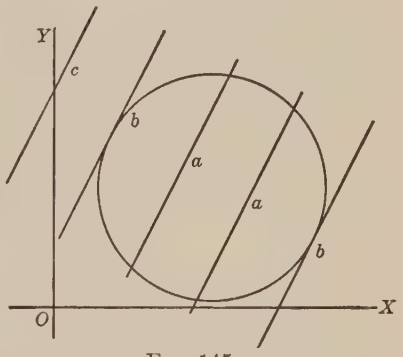


FIG. 145.

Now the intersections of this line and the circle may be (*a*) real and distinct, (*b*) real and coincident, or (*c*) imaginary, depending on the value of b. Eliminating y between (8) and (9) and collecting terms, we find

$$(1 + m^2)x^2 + 2bmx + (b^2 - r^2) = 0.$$

In order that the line be tangent to the circle, the roots of the equation must be equal; hence the discriminant of this equation must be zero, that is,

$$4b^2m^2 - 4(1 + m^2)(b^2 - r^2) = 0.$$

Solving for b,

$$b = \pm r\sqrt{1 + m^2}.$$

Hence for all real values of m the equation

$$y = mx \pm r\sqrt{1 + m^2} \tag{10}$$

represents the two tangents of slope m to the circle

$$x^2 + y^2 = r^2.$$

NOTE. If the equation of the circle is

$$(x - a)^2 + (y - b)^2 = r^2,$$

then the equation of the tangents is

$$y - b = m(x - a) \pm r\sqrt{1 + m^2}.$$

EXAMPLE. Find the equation of a line which is parallel to the line $y = 3x + 7$ and tangent to the circle $x^2 + y^2 = 25$.

SOLUTION. Here $m = 3$ and $r = 5$. Hence the required tangents are

$$y = 3x \pm 5\sqrt{1 + 9}.$$

234. Tangent to the Circle, Point Form.

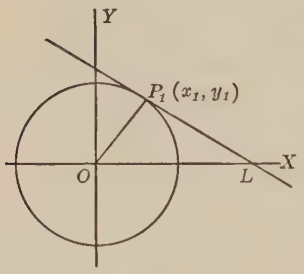


FIG. 146.

(*a*) *Center of the circle at the origin.* Let LP_1, Fig. 146, be the tangent to the circle

(11) $\qquad x^2 + y^2 = r^2$

at the point $P_1(x_1, y_1)$. Then the equation of the tangent, using the point slope form, is

(12) $\qquad y - y_1 = m(x - x_1).$

Taking the derivative of (11), we find

$$2x + 2y\frac{dy}{dx} = 0.$$

Hence the slope is

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope m at the point (x_1, y_1) on the curve is

$$m = -\frac{x_1}{y_1}$$

and the equation of the tangent at P_1 is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1). \tag{13}$$

Clearing of fractions and transposing, equation (13) becomes

$$xx_1 + yy_1 = x_1^2 + y_1^2.$$

Since $P_1(x_1, y_1)$ is a point on the circle,

$$x_1^2 + y_1^2 = r^2.$$

Simplifying,

$$xx_1 + yy_1 = r^2, \tag{14}$$

which is the standard form of the equation of the tangent.

(*b*) *Center of circle not at the origin.* Let the equation of the circle be

$$x^2 + y^2 + 2Dx + 2Ey + F = 0. \tag{15}$$

Taking the derivative, we find

$$2x + 2y\frac{dy}{dx} + 2D + 2E\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, the slope of the tangent at (x_1, y_1) is

$$\frac{dy}{dx} = -\frac{x_1 + D}{y_1 + E}.$$

Hence the equation of the tangent is

$$y - y_1 = -\frac{x_1 + D}{y_1 + E}(x - x_1). \tag{16}$$

Reducing, we obtain

$$xx_1 + yy_1 + 2D\frac{(x + x_1)}{2} + 2E\frac{(y + y_1)}{2} + F = 0, \tag{17}$$

which is the required equation of the tangent.

NOTE. The student should observe that the equation of the tangent is obtained by substituting xx_1 for x^2, yy_1 for y^2, $(x + x_1)/2$ for x, $(y + y_1)/2$ for y, the point (x_1, y_1) being on the circle. It will be found later that the above substitution holds for any equation of the second degree, the xy term being absent, and the point (x_1, y_1) being a point on the locus. If the point (x_1, y_1) is not on the given locus, the equation represents a line called the polar of the given point with respect to the locus.

EXAMPLE. Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

SOLUTION. Here $x_1 = 3$ and $y_1 = 4$. Substituting, we obtain

$$3x + 4y = 25,$$

the required equation.

235. Summary. The following summary is given for convenience of reference.

The equation of the circle, center at the origin, is

$$x^2 + y^2 = r^2.$$

The equation of the circle, center at (a, b), is

$$(x - a)^2 + (y - b)^2 = r^2.$$

The general equation of the circle is

$$x^2 + y^2 + Dx + Ey + F = 0.$$

The equation of the tangent to the circle $x^2 + y^2 = r^2$ at (x_1, y_1) is

$$(a) \qquad xx_1 + yy_1 = r^2.$$

The tangents with slope m are

$$(b) \qquad y = mx \pm r\sqrt{1 + m^2}.$$

EXERCISES

1. What is the equation of the tangents to the circle $x^2 + y^2 = 25$, if the slope of the tangent is 2?

2. Write the equation of the tangent to each of the following circles at the point indicated.

(*a*) $x^2 + y^2 = 100$, at (6, 8). (*d*) $x^2 + y^2 = 80$, at (4, 8).
(*b*) $x^2 + y^2 = 10$, at (1, −3). (*e*) $x^2 + y^2 = 17$, at the points where $x = 4$.
(*c*) $x^2 + y^2 = 20$, at (−2, 4). (*f*) $x^2 + y^2 = r^2$, at the point (x_1, y_1).

3. Write the equation of the tangent to each of the following circles at the point indicated.

(*a*) $x^2 + y^2 - 4x - 6y - 12 = 0$, at (5, 7).
(*b*) $x^2 + y^2 + 2x + 8y - 17 = 0$, at (2, 1).
(*c*) $x^2 + y^2 - 8x = 0$, at the points where $y = 2\sqrt{3}$.
(*d*) $x^2 + y^2 + Dx + Ey + F = 0$, at (x_1, y_1).

4. Write the equation of the tangents to each of the following circles, the slope of the tangents being as indicated. Find the points of contact.

(*a*) $x^2 + y^2 = 16$, slope 2. (*c*) $x^2 + y^2 = 36$, slope 7/4.
(*b*) $x^2 + y^2 = 9$, slope 5/2. (*d*) $(x - 1)^2 + (y - 3)^2 = 25$, slope 1/4.

5. Given the circle $x^2 + y^2 = 45$ and the line $2x - y = 1$. Required the equations of the tangents to the circle which are parallel to the line.

6. Given the circle $(x - 2)^2 + (y - 3)^2 = 16$. Required the equations of the tangents parallel to the line $y = 3x - 4$.

7. The plan for a bridge shows a circular arch whose equation is $x^2 + y^2 = 909$. The approach is a straight line tangent to the circular arch at (3, 30). Will this line pass through the point (300, .3)?

8. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ which are inclined to the axis of x at the following angles:

(*a*) 45°, (*b*) 30°, (*c*) 135°, (*d*) 60°.

9. A line is drawn tangent to the circle $x^2 + y^2 = 82$, at the point (1, 9). Find its length, that is, the distance from the point of tangency to the point of crossing the x-axis.

10. A point is moving on the circumference of the circle $x^2 + y^2 = 25$. Find its direction of motion when the abscissa of the point is 4 and its ordinate is positive.

11. Find by differentiation the slope of the tangent to the circle $x^2 + y^2 = 80$ at the point (8, 4). Also find the slope by some other method and compare results.

12. Find by differentiation the slope of the tangent to the circle $(x - 3)^2 + (y - 4)^2 = 25$ at the point (6, 8). Could you find the equation of this tangent by any other method?

13. A concrete pavement is to be 30 feet wide. A cross-section is to be the arc of a circle with a rise of one foot in the center. It is necessary to determine the radius in order to construct the form (templet) by which the concrete is laid. Find the radius.

236. Conics. The locus of a point which moves so that the ratio of its distances from a fixed point and from a fixed line is constant is called a ***conic.***

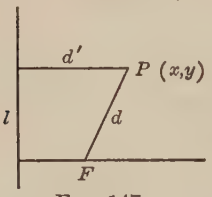


Fig. 147.

Let the moving point be designated by P, the fixed point by F, the fixed line by l, the distance FP by d, the distance of P from l by d', and the constant ratio d/d' by e. The fixed point F is called the ***focus*** of the conic, the fixed line l is called the ***directrix,*** and the constant ratio e is called the ***eccentricity.***

Note. In § 230 we have seen that conics were first studied by the methods of elementary geometry as plane sections of a right circular cone. In the following paragraphs we shall study conics by means of coordinates, a method which is much simpler than the geometric method of the Greeks.

237. Classification of Conics. The form of a conic depends on the value of the ratio d/d'.

If $d = d'$, then $e = 1$, and the conic is a ***parabola.***
If $d < d'$, then $e < 1$, and the conic is an ***ellipse.***
If $d > d'$, then $e > 1$, and the conic is a ***hyperbola.***
The circle is a special case of the ellipse in which $e = 0$.

238. The Parabola. The parabola is frequently found in the arch of a bridge or ceiling, orbits of comets, the curve of the supporting surface of an airplane, the curve of the propeller blades, and in graphs of scientific formulas. Its general characteristics can be seen from the equation

$y^2 = 4px$, which was derived in § 113. By definition, $FP = PQ$. That is,

$$\sqrt{(x - p)^2 + y^2} = x + p. \tag{18}$$

Therefore

$$y^2 = 4px. \tag{19}$$

1. This curve is symmetric with respect to the x-axis, has its vertex at the origin, its focus at $(p, 0)$, and the directrix is the line $x = -p$.

2. If p is positive, the curve lies to the right of the y-axis and extends indefinitely far.

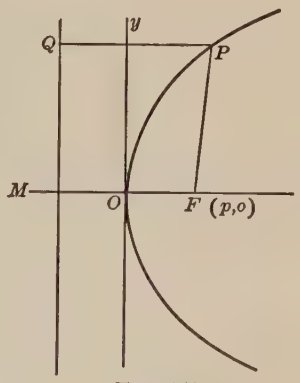


Fig. 148.

3. Its slope continually decreases. Differentiating, we have

$$2y\frac{dy}{dx} = 4p.$$

Hence the slope is

$$\frac{dy}{dx} = \frac{2p}{y}.$$

That is, the slope of the parabola $y^2 = 4px$ at any point is equal to $2p$ divided by the ordinate of the point. As y increases without limit, the slope approaches zero. Hence the curve tends to become parallel to the y-axis.

4. It has no maximum, no minimum, and no undulations.

5. The equation of a parabola whose vertex is at the point (a, b) and whose axis is parallel to the x-axis is

$$(y - b)^2 = 4p(x - a).$$

6. The general equation of the parabola whose axis is parallel to the x-axis is therefore of the form

$$By^2 + Dx + Ey + C = 0.$$

239. Focus on the y-Axis. When the y-axis is taken as the axis of the parabola, the vertex being at the origin, the equation has the form $x^2 = 4py$. The focus is at $(0, p)$. The equation of the directrix is $y = -p$. In this form the graph has either a maximum or a minimum.

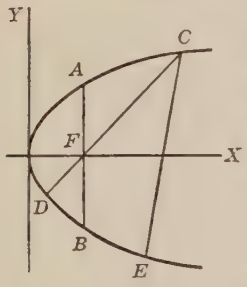


Fig. 149.

240. Chord. A line joining any two points on a conic is called a ***chord*** of the conic. If the chord passes through the focus, it is called a focal chord. The focal chord perpendicular to the axis is called the ***latus rectum***, and its length is called the ***focal width.***

A line from the focus of a conic to a point on the curve is called a ***focal radius.***

Note. In Fig. 149, AB, CD and CE are chords of this parabola. AB and CD are focal chords. AB is the latus rectum.

EXERCISES

Draw each of the following parabolas; find the coordinates of the (*a*) vertex, (*b*) focus; the equation of the directrix; the focal width.

1. $y^2 = 4x$. **3.** $y^2 = 16x$. **5.** $x^2 = 4y$. **7.** $x^2 = 8y$.

2. $y^2 = 12x$. **4.** $y^2 = -8x$. **6.** $x^2 = -20y$. **8.** $x^2 = 4ay$.

9. Given the equation $y^2 = 4px$, find the value of p under each of the following conditions.

(*a*) The parabola passes through (3, 6).
(*b*) The focal width is 12.
(*c*) The focus is at (6, 0). At $(-7, 0)$.
(*d*) The distance from the vertex to the focus is 7.

10. The arch of a bridge is a parabola with axis vertical. The horizontal span is 360 feet, and the vertex is 90 feet higher than the ends. Find the equation of the arch, the origin being at the vertex. Plot the curve. Find the slope of the arch at any point.

11. The roof of an auditorium is supported by parabolic arches. The horizontal span is 80 feet and the vertex of the arch is 16 feet higher

than the ends. Find the equation of the arch, the origin being at the vertex. Find the slope of the arch at the point (20, − 4). Find the distance of the arch below the vertex at intervals of 10 feet on either side of the center.

12. A cable of a suspension bridge is so loaded as to hang in the form of a parabola. The roadway, which is horizontal and 240 feet long, is supported by vertical rods attached to the cable, the longest being 60 feet and the shortest (from the vertex) 15 feet. Find the length of the supporting rods placed at intervals of 10 feet measured horizontally from the vertex. Find the slope of the cable at the points of attachment of the supporting rods.

13. Find the equation of a parabola, given that:

(*a*) The vertex is at (5, 3) and the focus at (8, 3).
(*b*) The vertex is at (0, − 4) and the focus at (0, − 8).
(*c*) The vertex is at (0, 6) and the directrix is the line $x = -6$.

14. The suspension cable of a level foot-bridge hangs in the form of a parabola. The horizontal span is 180 feet and the ends of the cable are 36 feet higher than the vertex. If the vertex is 24 feet above the water, find the height of the cable above the water at intervals of 10 feet measured horizontally from the vertex.

15. In the Brooklyn suspension bridge the horizontal distance MN between the towers is 1595 feet. The towers PM and QN are each 158 feet above the roadway MN. If the cable POQ is a true parabola, find the height of the cable above the roadway at intervals of 100 feet.

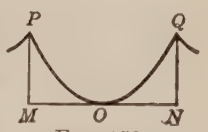


FIG. 150.

16. The parabolic reflecting surface of a telescope has a span of 24 inches and a depth of 2 inches. Find the equation of the parabola, and the focal width.

17. A tank whose cross section is a parabola with vertex downward is partly filled with liquid. The width of the tank 6 inches above the vertex is 24 inches. Find the depth of the liquid if the width of the surface of the liquid is 4 feet.

241. Tangent to the Parabola of Slope m. Let the equation of the parabola be

$$y^2 = 4px, \tag{20}$$

and let the equation of the tangent be

$$y = mx + k.$$

Since the tangent line PT may be defined as the limiting position approached by the secant line PQ as Q approaches P along the curve, the condition of tangency must be that of equal roots. Eliminating y between the equations, we have

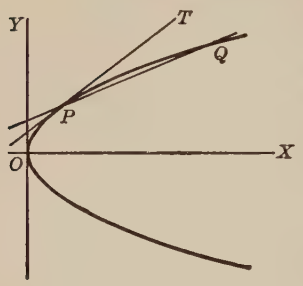


Fig. 151.

$$(mx + k)^2 = 4px,$$

or

$$m^2x^2 + (2mk - 4p)x + k^2 = 0.$$

Solving for x, we find

$$(21) \quad x = \frac{4p - 2mk \pm \sqrt{4m^2k^2 - 16pmk + 16p^2 - 4m^2k^2}}{2m^2}.$$

If the roots are equal, the discriminant (part under radical) must be zero, that is, $k = p/m$. Hence the equation of the tangent is

$$(22) \quad y = mx + p/m.$$

Example. Write the equation of the tangent to the parabola $y^2 = 8x$ if the tangent is parallel to the line $y = 3x - 11$.

Solution. $4p = 8$, hence $p = 2$, and $m = 3$. Substituting, $y = 3x + 2/3$, the required equation.

242. Tangent to the Parabola at (x_1, y_1). Since, by § 238, the parabola passes through (x_1, y_1) with the slope $2p/y_1$, the equation of the tangent is

$$(23) \quad y - y_1 = \frac{2p}{y_1}(x - x_1).$$

Clearing of fractions, we have

$$yy_1 - y_1^2 = 2px - 2px_1. \tag{24}$$

Since (x_1, y_1) is on the parabola, $y_1^2 = 4px_1$. Substituting and reducing, we may write the equation of the tangent in the form

$$yy_1 = 4p\frac{(x + x_1)}{2}. \tag{25}$$

EXAMPLE. The equation of the tangent to the parabola $y^2 = 4x$ at the point (9, 6) is

$$6y = \frac{4(x + 9)}{2}, \quad \text{or} \quad x - 3y + 9 = 0.$$

243. Recapitulation. The following data are given for convenience of reference.

Parabola along x-axis, vertex at origin: $y^2 = 4px$.
The eccentricity is $e = 1$.
The focus is the point $F = (p, o)$.
The directrix is the line $x = -p$.
The equation of the tangent at (x_1, y_1) is $yy_1 = 2p(x + x_1)$.
The equation of the tangent of slope m is $y = mx + p/m$.

EXERCISES

Write the equation of the tangent to each of the following curves at the point opposite.

1. $y^2 = 4x$, at (9, 6).
2. $y^2 = -2x$, at $(-2, 2)$.
3. $y^2 = 8x$, at (8, 8).
4. $x^2 = 4y$, at (4, 4).
5. $y^2 = 20x$, at (5, 10).
6. $y^2 = 4px$, at $(p, 2p)$.

Write the equation of the tangent to each of the following curves.

7. $y^2 = 4x$, tangent parallel to $y = 2x + 5$.
8. $y^2 = 8x$, tangent parallel to $3x - y = 7$.
9. $y^2 = 6x$, tangent perpendicular to $x + 5y = 10$.
10. $y^2 = 12x$, tangent makes an angle of 30° with x-axis.
11. $y^2 = 16x$, tangent makes an angle of 45° with x-axis.
12. $y^2 = 4x$, tangent parallel to line joining (1, 2) and (7, 14).

13. $y^2 = 6x$, tangent perpendicular to line joining (1, 0) and (5, 12).

14. $y^2 = 10x$, tangent parallel to a line which bisects the angle in the first quadrant.

15. $y^2 = 4x$, slope = 1/2.

16. $y^2 = 2x$, slope = 2/3.

17. Prove that the tangents at the extremities of the latus rectum of the curve $y^2 = 4px$ meet on the directrix.

18. Prove that the tangent at any point of the parabola $y^2 = 4px$, the perpendicular from the focus upon it, and the tangent at the vertex meet in a point.

19. Show that the focal width of the parabola $y^2 = 4px$ is $4p$.

FIG. 152.

20. A bridge truss has the form of a parabolic segment. The span AA' is 120 feet and the height OB, 20. Find the height of the perpendiculars $O'B'$, $O''B''$, etc., erected at intervals of 10 feet along $A'A$. Find the slope of the curve at the end of each of these ordinates.

21. A bridge truss has a parabolic segment 180 feet in span and 45 feet in height. Find the length of the ordinates erected at intervals of 15 feet, counted from the foot of the longest ordinate. Find the slope of the curve at the end of each of these ordinates.

22. Write the equation of the parabola which the family of lines $y = mx + 2/m$ envelops.

23. A projectile moves according to the following equations:

$$x = 400t, \qquad y = 200t - 16t^2.$$

Show that its path is a parabola.

SUGGESTION: Eliminate t from the equations.

244. The Ellipse. The ellipse is very common in nature, and has an extensive use in the arts and sciences. The meridians of the earth are ellipses. The orbit of the earth (or any planet) is an ellipse, the sun being at one focus. The Colosseum and the Arena at Rome were famous elliptical structures; the former being 615 feet long by 510 feet wide. The latter was 281 feet long and 177 feet wide.

The ellipse is found in (*a*) bridge construction (elliptical arches), (*b*) in the intersection of circular cylinders in constructing pipes and tunnels, (*c*) in auditoriums and whispering galleries. In the latter, faint sounds originating at one focus are reflected to the other and can be heard there though inaudible between these points.

On machines such as the hay press, planers, punches, and the like, the operation of pressing, planing, or punching needs to be slow and steady and the return motion rapid. Elliptical gears permit this combination of slow effective motion with quick return.

Shapely ovals, such as flower beds, mirrors, eye-glasses, portraits, etc., are frequently ellipses.

245. Definition and Terms. An ***ellipse*** may be defined as the locus of a point which moves so that the sum of its distances from two fixed points is constant. For another definition, see § 237. These two definitions can be shown to be equivalent. In Fig. 153 the two fixed points F and F' are called ***foci.*** The segment AA' on which the foci are located is called the ***major axis*** and is taken equal to $2a$; hence $OA = a$.

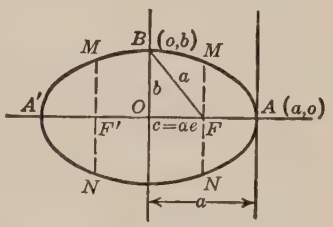


FIG. 153.

Similarly, the other axis BB', called the ***minor axis,*** is equal to $2b$; hence $OB = b$. The ends of the major axis A and A' are called the ***vertices*** of the ellipse; the point O is called the ***center;*** and the segments OA and OB are called the ***semi-axes.***

The distance FF' is taken equal to $2c$; hence OF or $OF' = c$.

246. Methods of Constructing an Ellipse. (*a*) *By continuous motion of a point on a drawing board.* Fasten two

tacks at points F and F' (the foci of the ellipse) and place around them a loop of string of length greater than FF'. If a pencil is placed in the loop and moved so as to keep the string taut, it will describe an ellipse. See Fig. 154. (*b*) *By points located by intersecting arcs.* Take two points F and F' as foci. Draw a straight line AA' of length $2a$, $2a$

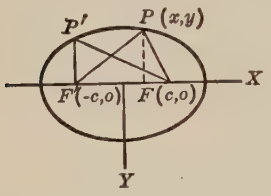


Fig. 154.

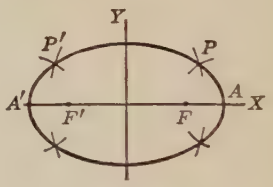


Fig. 155.

being greater than FF'. The points on the ellipse are the points of intersection of the arcs described by taking F and F' as centers, the sum of the focal radii FP and $F'P$ being kept constant and equal to $2a$. A smooth curve drawn through these points is the required locus. See Fig. 155.

247. Standard Equation, Center at Origin. Denote by $2c$ the distance between the foci, and by $2a$ the sum of the focal radii FP and $F'P$, the curve being symmetrically placed with respect to both axes (Fig. 154). By the distance formula

$$(26)\quad FP = \sqrt{(x-c)^2 + y^2}, \quad \text{and} \quad F'P = \sqrt{(x+c)^2 + y^2}.$$

Then

$$(27)\qquad \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

Transposing one radical, squaring and simplifying, we find

$$(28)\qquad cx + a^2 = a\sqrt{(x+c)^2 + y^2}.$$

Squaring again and simplifying, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{29}$$

Replacing $a^2 - c^2$ by b^2, we obtain the standard equation of the ellipse having its center at the origin:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{30}$$

This equation is often written in the form

$$b^2x^2 + a^2y^2 = a^2b^2. \tag{31}$$

248. Drawing the Ellipse. Solving the preceding equation, we find

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}. \tag{32}$$

From this equation it is easily seen that

1. The x intercepts are $+ a$, and $- a$.
2. The y intercepts are $+ b$, and $- b$.
3. The curve is symmetric with respect to both axes and the origin O.

As x increases from 0 to a, y decreases from b to 0, decreasing slowly when x is near zero but rapidly when x is near a. These facts are represented by the part of the curve in the first quadrant.

Since the curve is symmetric with respect to both axes the curve may be duplicated in each of the other quadrants.

EXAMPLE. Plot the graph of $x^2/25 + y^2/9 = 1$. Let x take on the successive values 0, 2, 3, 4, etc., and solve for the corresponding value of y.

x	0	± 2	± 3	± 4	± 5
y	± 3	± 2.7	± 2.4	± 1.8	0

Plotting these points and drawing a smooth curve through them, we obtain the graph, which will be found to be similar to Fig. 153.

249. Eccentricity. The ratio c/a is called the ***eccentricity,*** and is represented by the letter e. Since $c < a$, $e < 1$. See Fig. 153. Since $a^2 - c^2 = b^2$ (§ 247), we have

$$c = \sqrt{a^2 - b^2}. \tag{33}$$

But $e = c/a$. Substituting, we find

$$e = \frac{\sqrt{a^2 - b^2}}{a}. \tag{34}$$

When the foci are very near together, that is, when e is small, the ellipse differs but little from a circle.

It can be shown that e is also equal to the ratio d/d', which was represented by e in § 237.

250. Focal Distance. From the above equation the distance from the center to the focus (Fig. 153) is readily seen to be

$$OF = c = ae = \sqrt{a^2 - b^2}. \tag{35}$$

251. Focal Width. The chord MN, Fig. 153, drawn through either focus perpendicular to the major axis is called the latus rectum. Its length is called the ***focal width.*** To find this length, set $x = c$ in the standard equation, and solve for y. This gives

$$y = \pm\frac{b}{a}\sqrt{a^2 - c^2} = \pm\frac{b^2}{a}. \tag{36}$$

Hence the focal width $= 2b^2/a$.

EXAMPLE. An ellipse is drawn by the aid of a string 10 inches long, the ends being fastened at points 6 inches apart. Find (*a*) the equation of the ellipse, (*b*) the length of the major axis, (*c*) the length of the minor axis, (*d*) the eccentricity, (*e*) the focal width.

SOLUTION. Here $2a = 10$, and $2c = 6$. Hence $a = 5$, and $b = \sqrt{a^2 - c^2} = \sqrt{25 - 9} = 4$.

The equation is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

The length of the major axis is $2a = 10$. The length of the minor axis is $2b = 8$. The eccentricity

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{3}{5}.$$

The focal width is

$$\frac{2b^2}{a} = \frac{2 \times 16}{5} = 6\tfrac{4}{5}.$$

EXERCISES

1. Construct an ellipse in which $2a = 20$ and $FF' = 12$.

2. Construct an ellipse in which $2a = 10$ and $FF' = 8$.

3. Construct an ellipse whose semi-axes are 5 and 3, and write its equation. Use method (*b*), § 246.

4. In each of the following cases, write the equation of the ellipse. Draw the graph.

(*a*) $a = 6, b = 5$.
(*b*) $a = 4, b = 3$.
(*c*) $a = 5, b = 2$.
(*d*) $a = 8, b = 4$.
(*e*) $a = 10, b = 6$.
(*f*) $a = 15, b = 12$.
(*g*) $a = 20, b = 5$.
(*h*) $a = 25, b = 18$.

5. Draw the graph for each of the following ellipses. Find the foci, eccentricity, and focal width.

(*a*) $\frac{x^2}{25} + \frac{y^2}{16} = 1$.
(*b*) $4x^2 + 25y^2 = 100$.
(*c*) $4x^2 + 9y^2 = 36$.
(*d*) $9x^2 + 25y^2 = 225$.
(*e*) $3x^2 + 4y^2 = 24$.
(*f*) $x^2 + 4y^2 = 16$.

6. If the major axis is twice as long as the minor axis, find the eccentricity of the ellipse.

7. Solve each of the following pairs of simultaneous equations graphically and by one other method.

(*a*) $\begin{cases} \frac{x^2}{36} + \frac{y^2}{9} = 1, \\ x + 2y = 6. \end{cases}$

(*b*) $\begin{cases} \frac{x^2}{32} + \frac{y^2}{18} = 1, \\ x^2 + y^2 = 25. \end{cases}$

(*c*) $\begin{cases} \frac{x^2}{25} + \frac{y^2}{4} = 1, \\ y^2 = 4x. \end{cases}$

(*d*) $\begin{cases} \frac{x^2}{25} + \frac{y^2}{9} = 1, \\ x - y = 1. \end{cases}$

(*e*) $\begin{cases} \frac{x^2}{25} + \frac{y^2}{16} = 1, \\ x^2 + y^2 = 16. \end{cases}$

(*f*) $\begin{cases} \frac{x^2}{25} + \frac{y^2}{16} = 1, \\ x^2 + y^2 = 25. \end{cases}$

8. An ellipse has its center at (0, 0), and its major axis along XX'. Find its equation having given

(*a*) the longest diameter 16 and the shortest 10.

(*b*) the longest diameter 30 and the shortest 14.

9. An ellipse has its center at (0, 0), and the major axis along XX'. Find its equation having given

(*a*) foci ($\pm$ 4, 0), vertices at ($\pm$ 6, 0).

(*b*) foci ($\pm$ 5, 0), vertices at ($\pm$ 8, 0).

(*c*) length of minor axis 6, and foci at ($\pm$ 4, 0).

(*d*) points (4, 3) and (5, $3/\sqrt{2}$) on the ellipse.

10. An ellipse is drawn with a string 40 inches long, the ends being 20 inches apart. Find its equation.

11. Using pins 10 inches apart, and a loop of string whose total length is 26 inches, what a and b will the equation have?

12. A bridge has an arch in the form of half an ellipse in which $2a = 100$, and $b = 20$. Find the height of the arch at intervals of 10 ft. from its center along the major axis.

13. It is desired to lay out an elliptical flower bed 20 ft. long and 16 ft. wide. How far apart should the foci be? What length of string should be used?

14. A carpenter wishes to construct a frame to support the brickwork while laying up an arch over a gate. The arch is to be a semi-ellipse, 48 inches long and 15 inches high. Show how to locate the foci. Draw the ellipse to scale.

15. The arches of London bridge are semi-ellipses. The central arch has a span of 152 ft. and a height of 37.8 ft. Draw the arch to scale and find its equation.

16. A bridge has a semi-elliptical arch of span 120 ft. and height 24 ft. The level roadway is 4 feet above the center of the arch. How far is the roadway above the arch at intervals of 10 ft. measured along the roadway from a point above the center of the arch?

17. The orbit of the earth is an ellipse, the major axis being approximately 186 000 000 miles, and the distance between the foci 3 000 000 miles. Find its equation. Draw this ellipse to some convenient scale.

18. Show that the distance from one end of the minor axis of the ellipse to either focus is equal to the semi-major axis..

19. Find the equation of the locus of a point which moves so that the sum of its distances from the points (3, 0) and (− 3, 0) is constantly equal to 10. What conic section is this locus?

20. A point moves so that its distance from (4, 0) is equal to one-half its distance from the y-axis. What conic section is it? Find its equation.

252. Auxiliary Circles. The circle constructed on the major axis of the ellipse as a diameter is called the ***major auxiliary circle.*** Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{37}$$

$$\text{or} \qquad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

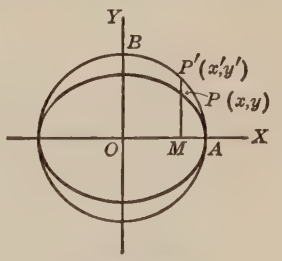


Fig. 156.

The equation of the circle is

$$x'^2 + y'^2 = a^2, \tag{38}$$

$$\text{or} \qquad y' = \pm\sqrt{a^2 - x^2},$$

since $x = x'$. Hence

$$\frac{y}{y'} = \frac{b}{a}. \tag{39}$$

That is, any ordinate of the ellipse is to the corresponding ordinate of the major auxiliary circle as b is to a.

Similarly, the circle constructed on the minor axis of the ellipse as a diameter is called the *minor auxiliary circle.*

253. Area of an Ellipse. The ordinate of an ellipse is to the corresponding ordinate of the major auxiliary circle as b is to a. It can be shown that the same relation exists between the areas. That is,

$$\frac{\text{Area of ellipse}}{\text{Area of major auxiliary circle}} = \frac{b}{a}. \tag{40}$$

But the area of the circle is πa^2. Hence the area of an ellipse whose semi-axes are a and b is

$$\frac{b}{a}(\pi a^2) = \pi ab.$$

It should be observed that the circle is a special case of the ellipse in which

$$a = b = r, \qquad \text{or} \qquad \pi ab = \pi aa = \pi r^2.$$

The circumference of the ellipse is approximately $\pi(a + b)$, provided its eccentricity is not too large.

EXERCISES

1. Every ordinate of a circle of diameter 36″ was reduced in the ratio 2 to 1. Construct the ellipse and write its equation.

2. Given the equation $x^2/49 + y^2/16 = 1$, draw the major and minor auxiliary circles. Locate a series of points on the ellipse. Draw the ellipse and find (*a*) its area, (*b*) its perimeter (approximately).

3. Find the area of an elliptic photograph 6″ long and $3\frac{1}{2}$″ wide.

4. The Colosseum of Rome was an elliptical structure 615 feet long by 510 feet wide. Find approximately its perimeter. How many acres did it cover? Draw the ellipse to scale.

5. The Arena of Rome was an ellipse 281 ft. by 177 ft. Compare its (*a*) perimeter, (*b*) area with that of the Colosseum.

6. Find the volume of a cone whose height is 24 inches and whose base is an ellipse, the major axis being 18 inches and minor axis 12 inches long.

7. A circular log 12 inches in diameter is cut by a plane making an angle of 30° with its axis. Find the equation and area of the section formed.

8. Every ordinate of a circle of radius 20 was reduced in the ratio 5 to 3. Find (*a*) the axes of the resulting ellipse, (*b*) the equation, (*c*) the approximate area.

9. The ground plan of an auditorium is elliptical in shape. The length of the major axis is 1300 feet and of the minor axis 1120 feet. Sound will travel from one focus to the other directly how much sooner than it will if reflected from the walls? Sound travels 1090 feet per second.

10. An ellipse which has its center at (5, 4) and its major axis parallel to the x-axis, is tangent to both the x-axis and the y-axis. Find its equation. Locate its foci.

11. The path of the earth about the sun is an ellipse with eccentricity .01677 = 1/60 nearly. If the semi-major axis of the earth's orbit is 92.9 million miles, find the length of the minor axis and the focal distance. Find the equation of the orbit and its approximate perimeter.

254. Tangent to the Ellipse, Slope Given. The tangent to the ellipse of slope m is obtained by eliminating y between the equations

$$(41) \qquad b^2x^2 + a^2y^2 = a^2b^2, \qquad \text{and} \qquad y = mx + k,$$

and then substituting for k its value obtained from the condition for equal roots of x. See § 241. Eliminating y between the equations and arranging the terms as a quadratic in x, we obtain

$$(42) \qquad (b^2 + a^2m^2)x^2 + 2a^2mkx + a^2(k^2 - b^2) = 0.$$

Solving for x, we find

$$(43) \qquad x = \frac{-a^2mk \pm \sqrt{k^2a^4m^2 - (b^2 + a^2m^2)(a^2k^2 - a^2b^2)}}{b^2 + a^2m^2}.$$

Putting the discriminant equal to zero (condition for equal roots and therefore the condition for a tangent) and solving for k,

$$k = \pm \sqrt{a^2m^2 + b^2}.$$

Hence if the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation of the tangents of slope m is

$$y = mx \pm \sqrt{a^2m^2 + b^2}. \tag{44}$$

NOTE. This method of obtaining the tangent of slope m applies to any curve given by an equation of the second degree.

255. Tangent to the Ellipse at the Point (x_1, y_1). Let the equation of the ellipse be in the form

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Taking the derivative, we find

$$2b^2x + 2a^2y\frac{dy}{dx} = 0.$$

Hence the slope of the tangent at (x_1y_1) is

$$-\frac{b^2x_1}{a^2y_1},$$

and the equation of the tangent is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1), \tag{45}$$

or

$$b^2xx_1 + a^2yy_1 = b^2x_1^2 + a^2y_1^2.$$

Since (x_1, y_1) is on the ellipse, we have

$$b^2x_1^2 + a^2y_1^2 = a^2b^2.$$

Hence the equation of the tangent may be written in the form

$$b^2xx_1 + a^2yy_1 = a^2b^2.$$

Dividing by a^2b^2, we may write the equation of the tangent in the form

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \tag{46}$$

NOTE. The intercept of the tangent on the x-axis is found by putting $y = 0$ and solving for x.

1. The x-intercept of the tangent is $x = \frac{a^2}{x_1}$.

2. Likewise the y-intercept of the tangent is $y = \frac{b^2}{y_1}$.

256. Normal. Since the normal is a line perpendicular to the tangent at the point of tangency (x_1, y_1), its slope will be the negative reciprocal of the slope of the tangent. Hence the equation of the normal is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1). \tag{47}$$

257. Ellipse, Center at (h, k). The equation of an ellipse referred to its axes $O'X'$ and $O'Y'$ is

$$\frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} = 1. \tag{48}$$

If the origin is taken as O, the coordinates (x, y) with reference to parallel axes through O have the values $x = x' + h$ and $y = y' + k$, where (h, k) is the center of the ellipse with reference to the new axes. Solving, $x' = x - h$ and $y' = y - k$.

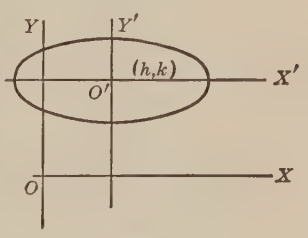


FIG. 157.

Substituting in (48), we obtain

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1. \tag{49}$$

This equation when cleared of fractions is of the form

$$Ax^2 + By^2 + Cx + Dy + E = 0. \tag{50}$$

258. Connection between Equations and Curves. From the study of the ellipse the student will see how useful coordinates are in the subject of geometry. The relation between curves and equations permits the study of geometrical properties of curves by means of their equations. These relations are useful in problems relating to distances, maxima and minima, slopes, rates, parallel and perpendicular lines, angles between lines, formulas for tangent and normal, various theorems of elementary geometry relating to loci, and the intersections of curves.

259. Recapitulation. For convenience of reference the following formulas with regard to the ellipse are restated here.

The standard equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The eccentricity is

$$e = \frac{\sqrt{a^2 - b^2}}{a}.$$

The foci are

$$F' = (-ae, 0), \qquad F = (ae, 0).$$

The tangent at $P_1(x_1, y_1)$ has the equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

The tangents with slope m have the equation

$$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

EXERCISES

1. In each of the following find the equation of the tangents which satisfy the given condition.

(*a*) $\frac{x^2}{16} + \frac{y^2}{9} = 1$, slope 2.

(*b*) $25x^2 + 36y^2 = 900$, slope 1/2.
(*c*) $4x^2 + 25y^2 = 100$, slope $-1/2$.
(*d*) $16x^2 + 64y^2 = 1024$, slope 4.
(*e*) $b^2x^2 + a^2y^2 = a^2b^2$, slope m.
(*f*) $9x^2 + 16y^2 = 144$, slope 0.

2. In each of the following equations, show that P is on the ellipse. Find the equations of the tangent and the normal at P.

(*a*) $x^2 + 4y^2 = 8$; P (2, 1).
(*b*) $9x^2 + y^2 = 1525$; P (13, 2).
(*c*) $5x^2 + 4y^2 = 21$; P $(-1, -2)$.
(*d*) $x^2 + 4y^2 = 164$; P (10, 4).
(*e*) $9x^2 + y^2 = 25$; P (1, 4).
(*f*) $3x^2 + 8y^2 = 83$; P (5, 1).
(*g*) $10x^2 + 3y^2 = 277$; P (5, 3).

3. A tangent is drawn to the ellipse $4x^2 + 25y^2 = 100$ at the point (3, 8/5). Find its intercept on (*a*) the x-axis, (*b*) the y-axis.

4. A tangent is drawn to the ellipse $16x^2 + 25y^2 = 400$ at the point whose abscissa is 4, and ordinate positive. Find its intercept on (*a*) the x-axis, (*b*) the y-axis. How far is the tangent from (*a*) the origin, (*b*) the right-hand focus?

5. Tangents are drawn to the ellipse and its major auxiliary circle at points having the same abscissa. Prove that they intersect on the x-axis.

6. If a, the major semi-axis of the ellipse, is kept constant, and b is allowed to vary, then the equation $b^2x^2 + a^2y^2 = a^2b^2$ represents many ellipses, one for each b. Show that the tangents to these ellipses at points having the same abscissa meet on the x-axis.

7. Find the equation of the tangent to the ellipse $9x^2 + y^2 = 82$ at (3, 1).

8. Find the points of intersections of each of the following pairs of curves.

(*a*) $x^2 + y^2 = 25$ and $12x^2 + 64y^2 = 768$.
(*b*) $y^2 = 4x$ and $16x^2 + 4y^2 = 320$.

9. Find the equations of the tangents to the ellipse $9x^2 + 4y^2 = 36$ which make an angle of 30° with the x-axis.

10. Find the slope of the ellipse $x^2/25 + y^2/9 = 1$ at the point $(3, 2\frac{2}{5})$.

11. Find the angle between the curves $4x^2 + y^2 = 32$ and $y = 2x$ at the point of intersection (2, 4).

12. An arch in the form of half an ellipse has a span of 20 feet and a height of 6 feet at the center. Find the equations of the tangent and the normal at the point on the ellipse where $x = 5$. Where $y = 2$. At the end of the latus rectum.

260. The Hyperbola. The hyperbola is commonly found in bridge construction, in orbits of comets, in locating invisible enemy guns in warfare, etc. A ***hyperbola*** is the locus of a point which moves so that the difference of its distances from two fixed points is constant.

For another definition see § 237. These two definitions can be shown to be equivalent.

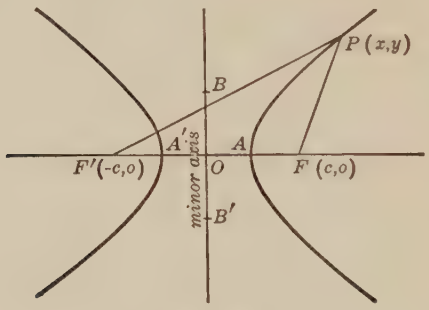


Fig. 158.

In Fig. 158 the two fixed points F and F' are called ***foci.*** The distance FF' is taken equal to $2c$; hence $OF = OF' = c$. The difference of the distances $F'P$ and FP is taken equal to $2a$. The segment AA' on which the foci are located is called the ***major axis*** and is equal to $2a$. Hence $OA = a$. Similarly BB', called the ***minor axis*** or the ***conjugate axis,*** is taken equal to $2b$, where $b = \sqrt{c^2 - a^2}$. Hence $OB = b$. The ends of the major axis A and A' are called the ***vertices*** of the hyperbola, the point O is called the ***center,*** and the segments OA and OB are called ***semi-axes.*** The line from a point on the curve to a focus is a ***focal radius.***

261. Methods of Construction. (*a*) *By continuous motion of a point.* Place thumb tacks at F' and F. Tie a string to a pencil P and let one end of the string pass beneath F and then both ends over F', Fig. 159. Adjust the string so that $F'P$ exceeds FP by $2a$. By holding the strings below F' and drawing them in or letting them out together, the point P will trace a hyperbola, since for every point on the curve $F'P - FP = 2a$.

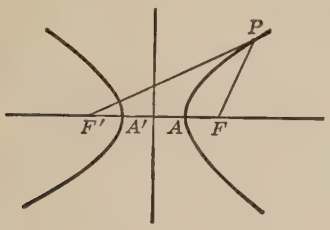


Fig. 159.

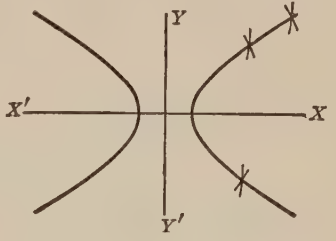


Fig. 160.

(*b*) *By separate points.* Any number of points on the hyperbola can be found by describing arcs having centers at the fixed points F and F', and with radii FP and $F'P$ which differ by a fixed amount, $2a$. Two separate curves are obtained according as we choose FP or $F'P$ the larger. The two curves taken together are called the hyperbola.

262. Equation of the Hyperbola, Center at the Origin. Denote by $2c$ the distance between the foci and by $2a$ the difference of the focal radii, and let the curve be symmetrically placed with reference to both axes. By the distance formula

$$(51)\quad F'P = \sqrt{(x+c)^2 + y^2} \quad \text{and} \quad FP = \sqrt{(x-c)^2 + y^2}.$$

Then from the definition of a hyperbola,

$$(52)\qquad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a.$$

Transposing one radical, squaring and simplifying,

$$cx - a^2 = a\sqrt{(x - c)^2 + y^2}. \tag{53}$$

Squaring again and simplifying, we find

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1. \tag{54}$$

Replacing $c^2 - a^2$ by b^2, we obtain the general equation of the hyperbola, having its center at the origin, namely,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{55}$$

This equation is often written in the form

$$b^2x^2 - a^2y^2 = a^2b^2. \tag{56}$$

263. Drawing the Hyperbola. Solving the above equation, we find

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}. \tag{57}$$

From this equation it is easily seen that

(1) The x-intercepts are a and $-a$.
(2) When x is 0, y is imaginary; hence the curve does not cross the y-axis.
(3) When $x = \pm a$, $y = 0$. If x is numerically greater than a, the value of y is real and is either positive or negative. As x increases without limit, y increases without limit.
(4) The curve is symmetric with respect to both axes and the origin.

EXAMPLE. Plot the graph of

$$\frac{x^2}{25} - \frac{y^2}{9} = 1.$$

Solving for y,

$$y = \pm \tfrac{3}{5}\sqrt{x^2 - 25}.$$

Now let x take on the successive values 5, 6, 7, 8, 9, etc., and solve for the corresponding value of y.

x	± 5	± 6	± 7	± 8	± 9
y	0	± 1.99	± 2.94	± 3.75	± 4.49

Plotting these points and drawing a smooth curve through them, we obtain the graph, Fig. 161.

264. Asymptotes. The diagonals OC and OC' of the rectangle on the axes are called ***asymptotes*** of the hyperbola. The curve approaches closer and closer to these lines as the moving point recedes from the vertices. Obviously the equations of the asymptotes are

$$y = \frac{b}{a}x, \quad \text{and} \quad y = \frac{-b}{a}x.$$

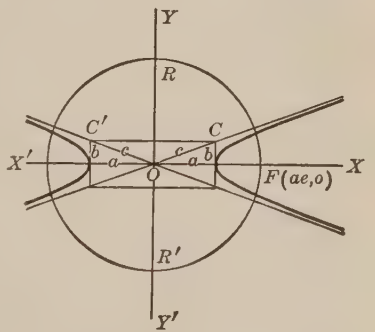


Fig. 161.

Hence, if the equation of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ is given, a simple way to draw the hyperbola is to draw the asymptotes, locate the vertices, and sketch the curve.

The equations of the asymptotes in the example of § 263 are

$$y = \pm\frac{3x}{5}.$$

265. Eccentricity. The ratio c/a is called the eccentricity and is represented by the letter e. Since $c > a$, $e > 1$. Since $c^2 - a^2 = b^2$, by § 262, we have

$$c = \sqrt{a^2 + b^2}. \tag{58}$$

But $e = \frac{c}{a}$. Substituting, we find

$$e = \frac{\sqrt{a^2 + b^2}}{a}. \tag{59}$$

266. Focal Distance. From the above equation the distance from the center to the focus is readily seen to be

$$OF = c = ae = \sqrt{a^2 + b^2}. \tag{60}$$

EXERCISES

1. Using method (*a*), § 261, construct several hyperbolas.

2. Using method (*b*), § 261, construct a hyperbola in which $2a = 6$, and $F'F = 10$.

3. Write the equation of the hyperbola, having given

(*a*) $a = 6, b = 3$.
(*b*) $a = 7, b = 5$.
(*c*) $a = 9, b = 4$.
(*d*) $a = 10, b = 6$.
(*e*) $a = 12, b = 8$.
(*f*) $a = 9, b = 1$.

4. Draw the graph of each of the following hyperbolas.

(*a*) $9x^2 - 16y^2 = 144$.
(*b*) $25x^2 - 36y^2 = 900$.
(*c*) $4x^2 - 49y^2 = 196$.
(*d*) $x^2 - 16y^2 = 16$.

5. A hyperbola has its center at (0, 0) and its axes along XX' and YY'. Find its equation, having given

(*a*) Foci $(\pm 5, 0)$, vertices at $(\pm 4, 0)$.
(*b*) Foci $(\pm 10, 0)$, vertices at $(\pm 6, 0)$.

6. Find the semi-axes, eccentricity, and coordinates of the foci in each of the following cases.

(*a*) $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
(*b*) $9x^2 - 25y^2 = 225$.
(*c*) $25x^2 - 144y^2 = 3600$.
(*d*) $9x^2 - 16y^2 = 144$.
(*e*) $x^2 - 4y^2 = 4$.
(*f*) $b^2x^2 - a^2y^2 = a^2b^2$.

7. Solve each of the following pairs of simultaneous equations graphically and by one other method.

(*a*) $\begin{cases} 9x^2 - 25y^2 = 225, \\ x^2 + y^2 = 36. \end{cases}$
(*b*) $\begin{cases} 36x^2 - 64y^2 = 2304, \\ y^2 = 4x. \end{cases}$
(*c*) $\begin{cases} \frac{x^2}{16} - \frac{y^2}{4} = 1, \\ y = 2x - 1. \end{cases}$
(*d*) $\begin{cases} 100x^2 - 36y^2 = 3600, \\ x^2 + y^2 = 64. \end{cases}$

8. Find the equation of the locus of a point which moves so that the difference of its distances from the points $(-5, 0)$ and $(5, 0)$ is constant and equal to 8.

SUGGESTION. Take as the x-axis the line joining the fixed points, the origin being midway between them. Then $2c = 10$, and $2a = 8$. Hence $a = 4$ and $b = \sqrt{c^2 - a^2}$.

9. Two points are 52 yards apart. Find the locus of all points 48 yards farther from one than from the other.

10. Show that the tangents to the hyperbola which have the slope m are $y = mx \pm \sqrt{a^2m^2 - b^2}$. SUGGESTION. See method of § 254.

11. A circle moves and changes in size so as to remain always tangent to two fixed unequal circles. Discuss the locus of the center of the moving circle.

267. Another Mechanical Construction. Pivot a ruler at the focus F'. A string whose length is $2a$ (less than the length of the ruler) is attached to the end of the ruler and to the focus F. A pencil which holds the string taut and against the ruler will, while the ruler revolves about F', trace one branch of the hyperbola. For every position of P, $F'P - FP = 2a$.

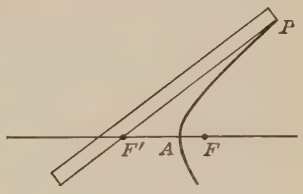


FIG. 162.

By reversing the position of string and ruler the other branch may be drawn.

268. Equilateral or Rectangular Hyperbola. If the hyperbola has its axes equal to each other, that is, $a = b$, the equation becomes

$$(61) \qquad \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1, \qquad \text{or} \qquad x^2 - y^2 = a^2.$$

This curve is called an equilateral hyperbola. It is also called a rectangular hyperbola since the asymptotes are perpendicular to each other. Both asymptotes pass

through the origin, one making an angle of 45° with the x-axis and the other an angle of 135°.

A form of the equation of the rectangular hyperbola which is more convenient than (61) is obtained if the axes are rotated through an angle of 45°. This equation is

$$xy = \frac{\pm a^2}{2}, \qquad \text{or} \qquad xy = \pm k,$$

where k is a positive constant. If $\pm k$ is positive, x and y must be of like sign and the curve lie in the first and third quadrants. See Fig. 163.

269. To Construct an Equilateral Hyperbola when One Point and the Asymptotes are Given. Let OX and OY be

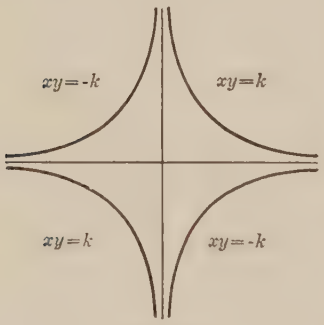


Fig. 163.

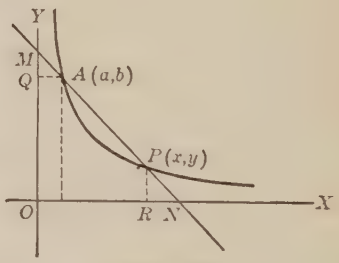


Fig. 164.

the asymptotes and A the given point. Draw any line through A meeting OX in N, and OY in M. Lay off $NP = AM$. Then P is a point on the required hyperbola.

Proof. From the equal triangles AMQ and NPR, $QM = RP$, $QA = RN$, and from the similar triangles QMA and OMN,

$$\frac{QM}{QA} = \frac{OQ + QM}{OR + RN}, \qquad \text{or} \qquad \frac{y}{a} = \frac{b + y}{x + a}.$$

Hence $xy = ab$.

270. Use of the Equilateral Hyperbola. The graph of the equilateral hyperbola is frequently useful in scientific work, and in making business calculations.

(*a*) Boyle's Law. This law states that the product of the volume of a gas at a given temperature and the pressure is constant, that is, $PV = C$. This condition is represented graphically by the branch in the first quadrant.

(*b*) Ohm's Law.

$$\text{Current in amperes} = \frac{\text{Electromotive force in volts}}{\text{Resistance in ohms}},$$

that is, $c = e/r$, or $cr = e$.

(*c*) In mensuration the area of a rectangle, triangle, ellipse, etc., is expressed by an equation of the form $xy = k$.

(*d*) Engravers use a chart like Fig. 165 from which the areas of half tones or zinc etchings are read. The plate is placed on the chart, as shown in Fig. 165. If the corner C

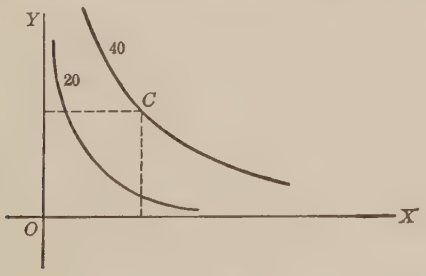


Fig. 165.

falls anywhere on the curve $xy = 40$, then the area of the plate is 40 square inches, and the price is read from a table.

EXERCISES

1. Draw a hyperbola, using the method of § 267, having given $F'F = 10$ and $2a = 6$. Write its equation.

2. Draw a hyperbola, using the method of § 267, having given $F'F = 26$ and $2a = 24$. Write its equation.

3. Draw the graph of the equilateral hyperbola $x^2 - y^2 = 16$.

4. What is the eccentricity of a hyperbola in which $a = b$?

5. What is the angle between the asymptotes of an equilateral hyperbola?

6. Draw the graph for (*a*) $xy = 4$, (*b*) $xy = 12$.

7. Find the equation of the locus of a point which moves so that the difference of its distances from the points (3, 3) and (− 3, 3) is 4.

8. Write the equation of the asymptotes for each of the following:

(*a*) $\frac{x^2}{25} - \frac{y^2}{9} = 1$. (*b*) $\frac{x^2}{36} - \frac{y^2}{16} = 1$. (*c*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

9. Draw the graph for (*a*) $vp = 24$, (*b*) $cr = 12$.

10. Construct the hyperbola $xy = 20$, using the method of § 269, one point on the hyperbola being (2, 10).

11. The arch of a bridge is a rectangular hyperbola 200 feet wide and 60 feet deep. Plot the curve. HINT. Use the equation $y^2 - x^2 = a^2$.

12. Draw the graph of each of the following hyperbolas.

(*a*) $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

(*b*) $\frac{x^2}{36} - \frac{y^2}{9} = 1$.

(*c*) $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

(*d*) $4x^2 - 25y^2 = 100$.

(*e*) $x^2 - y^2 = 25$.

(*f*) $xy = 12$.

(*g*) $\frac{x^2}{4} - y^2 = 1$.

(*h*) $x^2 - \frac{y^2}{9} = 1$.

13. If for a certain gas $v = 10$ cu. ft. when $p = 92$ lb., find the relation between v and p. Find p if $v = 20$ cu. ft.; $7\frac{1}{2}$ cu. ft.; 4 cu. ft. Use Boyle's Law.

14. Plot a graph showing the relation between c (current) and r (resistance) if $e = 110$ (volts). Use Ohm's Law.

15. Draw the graphs of

(*a*) $y = \frac{2}{x}$. (*b*) $y = \frac{-4}{x}$. (*c*) $yx = 6$.

16. Solve each of the following systems of simultaneous equations.

(*a*) $\begin{cases} xy = 12, \\ x + y = 7. \end{cases}$ (*c*) $\begin{cases} xy = 36, \\ x - y = 5. \end{cases}$ (*e*) $\begin{cases} xy = 20, \\ x^2 - y^2 = 41. \end{cases}$

(*b*) $\begin{cases} xy = 8, \\ \dfrac{x^2}{32} - \dfrac{y^2}{8} = 1. \end{cases}$ (*d*) $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{9} = 1, \\ \dfrac{x^2}{35} - \dfrac{y^2}{16} = 1. \end{cases}$ (*f*) $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{4} = 1, \\ y = \frac{1}{3}x. \end{cases}$

17. Draw the graph of $xy = k$ when $k = 12$, $k = 20$, $k = 40$. If the cost of half tones or zinc etchings is 32 cents per square inch, label each curve with the cost of any plate which fits it as shown in Fig. 165.

18. Rectangular copper plates of a given thickness weigh 2.84 oz. per square inch. Draw the curves for $xy = 4$, $xy = 10$, and label each curve with the weight of any rectangular plate which fits it.

271. Tangent to Hyperbola, Slope Given. The tangent to the hyperbola of slope m is obtained by eliminating y between the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{and} \qquad y = mx + k, \tag{62}$$

and then substituting for k the value of k obtained from the condition for equal roots of x. This gives

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{63}$$

It is left to the student to complete the proof. See § 254.

272. Tangent to Hyperbola at the Point (x_1, y_1). Let the equation of the hyperbola be in the form

$$b^2x^2 - a^2y^2 = a^2b^2. \tag{64}$$

Taking the derivative, we have

$$2b^2x - 2a^2y\frac{dy}{dx} = 0. \tag{65}$$

Hence the slope of the tangent at (x_1, y_1) is

$$\frac{b^2x_1}{a^2y_1}. \tag{66}$$

The equation of the tangent is

$$y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1), \tag{67}$$

or

$$b^2 x x_1 - a^2 y y_1 = b^2 x_1^2 - a^2 y_1^2. \tag{68}$$

But

$$b^2 x_1^2 - a^2 y_1^2 = a^2 b^2.$$

Dividing by $a^2 b^2$, we may write the equation of the tangent in the form

$$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1, \tag{69}$$

the equation of the tangent at (x_1, y_1).

NOTE. To write the equation of a tangent at the point of contact $P_1(x_1, y_1)$ to the locus of any equation of the second degree, substitute xx_1 for x^2, yy_1 for y^2, $(x_1 y + y_1 x)/2$ for xy, $(x + x_1)/2$ for x, and $(y + y_1)/2$ for y, in the given equation.

273. Normal to the Hyperbola. The equation of the normal to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point $P_1(x_1, y_1)$ is

$$y - y_1 = \frac{-a^2 y_1}{b^2 x_1}(x - x_1). \tag{70}$$

274. Generalized Standard Equation of the Hyperbola. As in the case of the circle and ellipse, the equation of the hyperbola can be generalized so as to apply to any hyperbola with axes parallel to the x- and y-axes. Let the center of the hyperbola be at (h, k). Then the equation is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1, \tag{71}$$

where a and b are the semi-major and semi-minor axes, respectively. The proof is left to the student. See § 257.

275. Recapitulation. For convenience of reference, the following data with regard to the hyperbola are re-stated here. The standard equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The eccentricity is

$$e = \sqrt{\frac{a^2 + b^2}{a}}, \qquad \text{and} \qquad e > 1.$$

The foci are

$$F' = (-ae, 0), \qquad F = (ae, 0).$$

The tangent at P_1 is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

The tangents with slope m are

$$y = mx \pm \sqrt{a^2m^2 - b^2}.$$

The equation of the asymptotes is

$$y = \pm \frac{bx}{a}.$$

The equation of the equilateral hyperbola is

$$xy = \pm k.$$

EXERCISES

1. Find the equation of the tangent at the point indicated to each of the following hyperbolas.

(*a*) $x^2 - 6y^2 = 19$, at (5, 1).
(*b*) $2x^2 - 4y^2 = 16$, at (4, 2).
(*c*) $5x^2 - 3y^2 = 42$, at (3, 1).
(*d*) $6x^2 - 8y^2 = 88$, at (6, 4).
(*e*) $b^2x^2 - a^2y^2 = a^2b^2$, at (x_1y_1).
(*f*) $3x^2 - 5y^2 = 88$, at (6, 2).

2. Find the equations of the tangents to each of the following hyperbolas which satisfy the conditions indicated.

(*a*) $4x^2 - 9y^2 = 36$, slope 2.
(*b*) $25x^2 - 49y^2 = 1225$, slope 3.
(*c*) $\frac{x^2}{16} - \frac{y^2}{7} = 1$, slope $\frac{1}{2}$.
(*d*) $\frac{x^2}{36} - \frac{y^2}{4} = 1$, slope $\frac{1}{3}$.
(*e*) $30x - 42y^2 = 1260$, slope 1.
(*f*) $x^2 - y^2 = 4$, slope $\frac{1}{2}$.

3. Find the equation of the tangent to each of the following curves at the point of contact (x_1, y_1).

(*a*) $x^2 + y^2 = 25$.
(*b*) $5x^2 + 4y^2 = 40$.
(*c*) $3x^2 - y^2 = 48$.
(*d*) $y^2 = 4x$.
(*e*) $x^2 = 4y$.
(*f*) $x^2 + y^2 = 17$.
(*g*) $x^2 + y^2 + 2x = 10$.
(*h*) $3x^2 + 4y^2 + 4x - 6y = 12$.
(*i*) $xy = 16$.
(*j*) $3x^2 + 4y^2 - 5xy = 20$.
(*k*) $2x^2 + 3y^2 + 2xy + 4x + 8y + 6 = 0$.
(*l*) $ax^2 + by^2 + cxy + dx + ey + f = 0$.

4. Find the equations of the tangents at the point or points whose abscissa is the number set opposite, for each of the following curves.

(*a*) $x^2 + y^2 = 40$, 2.
(*b*) $5x^2 + 4y^2 = 61$, 3.
(*c*) $8x^2 - y^2 = 127$, 8.
(*d*) $y^2 = 8x$, 2.
(*e*) $4x^2 + 9y^2 = 136$, 5.
(*f*) $x^2 = 4y$, 4.

5. Find the tangents to the circle $x^2 + y^2 = 25$ which make an angle of 45° with the x-axis.

6. Find the tangents to the parabola $y^2 = 4x$ which make an angle of 60° with the x-axis. 30° with the x-axis.

7. What are the equations of the tangents to the hyperbola $16x^2 - 25y^2 = 400$ which cross the x-axis at an angle of 60°?

8. Find the slope of the tangent to the curve $x^2/36 - y^2/16 = 1$ at the point (x_1, y_1).

9. Given the equation $9x^2 - 16y^2 = 144$. Find the distance of the tangent at one end of the latus rectum from the nearer focus.

276. Translation of Coordinate Axes. The equation of a curve can be simplified sometimes by moving the axes to the right or left, up or down, without rotating them. The axes are then said to be translated from the first to the second position.

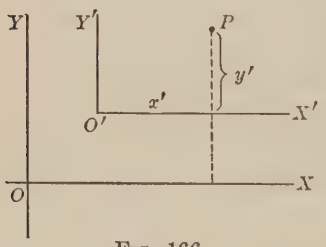


Fig. 166.

Let the new origin O' be (h, k), Fig. 166, and the coordinates of the point P with respect to OX and OY be (x, y) and with respect to $O'X'$ and $O'Y'$ be (x', y'). Then from the figure we readily obtain the equations of translation

$$x = x' + h, \qquad \text{and} \qquad y = y' + k.$$

Hence the equation of the curve with reference to the new axes is readily obtained by replacing each x in the original equations by $x' + h$ and each y by $y' + k$.

EXAMPLE 1. Given the equation

$$x^2 + y^2 - 6x - 10y + 18 = 0,$$

to find the transformed equation when the axes are translated to a new origin at (3, 5).

SOLUTION. Here $h = 3$, $k = 5$; hence the equations of translation are $x = x' + 3$, and $y = y' + 5$.

Substituting in the given equation and dropping the primes,

$$(x + 3)^2 + (y + 5)^2 - 6(x + 3) - 10(y + 5) + 18 = 0,$$

or

$$x^2 + y^2 = 16.$$

EXAMPLE 2. The equation of the parabola $(y - 1)^2 = 4(x - 3)$ becomes $y^2 = 4x$ when the axes are moved 3 units to the right and 1 unit upward.

EXAMPLE 3. The equation of the hyperbola $(x - 4)^2/36 - (y - 5)^2/9 = 1$ becomes $x^2/36 - y^2/9 = 1$ when referred to parallel axes through (4, 5).

277. Conics in General. Any equation of the second degree in rectangular coordinates represents a conic. The equation

$$ax^2 + hxy + by^2 + 2cx + 2dy + e = 0 \tag{72}$$

is called the ***general equation of the second degree*** in x and y because it contains all possible terms in x and y of the second and lower degrees. This equation has six terms. If we divide by any non-vanishing coefficient, say e, the equation becomes

$$px^2 + qxy + vy^2 + sx + ty + 1 = 0, \tag{73}$$

which involves five constants. Hence a conic is determined by five independent conditions. For example, the equation of the conic can in general be determined if the coordinates of five points through which it passes are known.

By rotating the axes through a properly chosen angle, the xy term can be eliminated and the equation becomes

$$Ax^2 + By^2 + Cx + Dy + E = 0, \tag{74}$$

which always represents a conic section, real or imaginary, whose axes are parallel to the coordinate axes.

1. Straight Line. If $A = B = 0$, the equation represents a straight line.

2. The Circle. If $A = B \neq 0$, the equation represents a circle and can be reduced to the standard form

$$x^2 + y^2 = r^2$$

by moving the origin to the center of the circle.

3. The Parabola. If either A or B equals zero, but not both, the equation in general represents a parabola and can be reduced to one of the standard forms

$$y^2 = 4px \qquad \text{or} \qquad x^2 = 4py$$

by moving the origin to the vertex. In special cases this parabola degenerates into a pair of parallel straight lines, real or imaginary; for instance, if $B = 0$ and $D = 0$.

4. The Ellipse. If neither A nor B is zero, and they have like signs, the curve is in general an ellipse and the equation may be reduced to the standard form. In exceptional cases this ellipse degenerates into a point, or an imaginary locus. If $A = B$, the ellipse becomes a circle.

5. The Hyperbola. If neither A nor B is zero, and they have unlike signs, the curve is in general a hyperbola and the equation may be reduced to the standard form by moving the origin to the center of the conic. In exceptional cases this hyperbola degenerates into a pair of real intersecting lines.

The circle, ellipse and hyperbola have centers and therefore are called ***central conics*** while the parabola is said to be *non-central.*

EXERCISES

Reduce each of the following equations to a standard form and draw the conic represented by it.

1. $x^2 + y^2 - 2x - 4y = 20$.

2. $x^2 + y^2 + 8x + 6y = 11$.

3. $x^2 + y^2 - 2x - 10y = -1$.

4. $(y - 1)^2 = 4(x - 4)$.

5. $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{16} = 1$.

6. $\frac{(x-4)^2}{36} - \frac{(y-1)^2}{16} = 1$.

7. A circle has its center at (5, 2) and its radius equal to 7. Write its equation.

8. A parabola has its vertex at (3, 5) and its focus at (6, 5). Write its equation.

9. An ellipse in which $a = 7$ and $b = 5$ has its center at $(3, -4)$ and its axes parallel to the coordinate axes. Write its equation.

10. An ellipse has its center at (6, 4) and is tangent to both axes. Write its equation.

11. A hyperbola has its center at (5, 6). Its semi-major axis is 4, its semi-minor axis is 3 and its axes are parallel to the coordinate axes. Find its equation and also the equation of the asymptotes.

12. Find the equation of a conic passing through the points $(1, -1)$, $(2, 0)$, $(1, 1)$, $(0, 0)$, and $(0, -1)$.

13. An ellipse has its foci at (4, 1) and (4, 7) and is tangent to the y-axis. Find its equation.

14. A point (x, y) moves so that the sum of its distances from $(-2, 0)$ and $(8, 0)$ is always 12. Find the equation of its path. What conic section is it?

15. Find the equations of the tangents to the circle $(x - 2)^2 + (y - 3)^2 = 25$, whose slope is 5.

CHAPTER XVI

SPACE OF THREE DIMENSIONS

278. Introduction. In applying algebra to the geometry of space, the position of a point may be determined by reference to three fixed planes which intersect in a given point. The fixed planes are called coordinate planes; the lines of intersection, the coordinate axes; and the point of intersection, the origin. Although there are many kinds of coordinates, rectangular, oblique, spherical, cylindrical, or polar coordinates, only the first will be discussed here.

279. Rectangular Coordinates. Let XOY, YOZ, and XOZ be three mutually perpendicular planes through the point O which intersect by pairs in the lines XX', YY', and ZZ'. These coordinate planes are designated as the xy-, yz-, and xz-planes; and the mutually perpendicular lines OX, OY, OZ as the x-, y-, and z-axes, respectively.

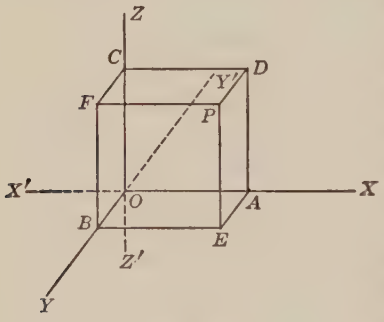


Fig. 167.

The coordinates of any point P are the respective distances of P from the coordinate planes measured, therefore, parallel to the axes. The x-coordinate is the distance from the yz-plane, the y-coordinate is the distance from the xz-plane, etc. For the point P in Fig. 167,

$$x = FP = BE = OA.$$
$$y = DP = AE = OB.$$
$$z = EP = AD = OC.$$

These rectangular coordinates serve completely to identify the position of the point P. For this reason the point is denoted by $P(x, y, z)$, or (x, y, z).

280. Convention of Signs. The coordinate planes divide all space into eight parts called octants. The octants are designated by $O\text{-}XYZ$, $O\text{-}X'YZ$, etc. The x-coordinate is positive or negative according as P lies to the right or left of the yz-plane. Similarly, y is positive or negative according as P lies in front or back of the xz-plane, and z is positive or negative according as P lies above or below the xy-plane.

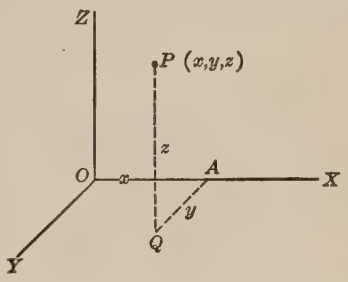


FIG. 168.

The octant in which the coordinates (x, y, z) of a point are all positive is called the first octant. It is not necessary to assign to the other seven octants a specific order.

EXERCISES

1. What are the coordinates of the origin?

2. Plot the following sets of points.

(*a*) (4, 3, 2), (6, 5, 3), (8, 1, 6).
(*b*) (− 4, 3, 5), (− 3, 7, 2), (− 4, 6, 1).
(*c*) (5, − 3, 7), (6, − 2, 5), (7, − 5, 4).
(*d*) (2, 3, − 4), (6, 3, − 5), (7, − 2, − 3).
(*e*) (0, 0, 3), (0, 5, 0), (7, 0, 0).

3. Where can a point be if $x = 0$? $y = 0$? $z = 0$?

4. What is the z-coordinate of any point in the xy-plane?

5. What are the x and y coordinates of any point on the z-axis?

6. A rectangular solid lies in the first octant with three faces in the coordinate planes. If its dimensions are 6, 4, and 3, respectively, find the coordinates of its vertices.

7. In which of the coordinate planes are the points (3, 4, 0), (3, 0, 7), (4, 0, − 1), (0, 3, 8), (0, − 4, 3)?

8. Show that the following points are on one of the coordinate axes. State which axis. (5, 0, 0), (0, 0, − 8), (0, 0, 4), (6, 0, 0), (0, 8, 0), (0, − 3, 0).

9. Where are all the points for which $z = -2$? $x = 5$? $y = 4$?

10. The coordinates of a point are (3, 5, 8). State the coordinates of a point which is symmetric with respect to the xz-plane, the xy-plane, the yz-plane.

11. Find the point symmetric to each of the following points with respect to the origin. (3, 4, 6), (− 3, 4, − 7), (5, − 4, 3).

12. Show that (a, b, c) and $(-a, -b, -c)$ are symmetric with respect to the origin.

281. Distance between Two Points. The distance between any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by the formula

$$(1) \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

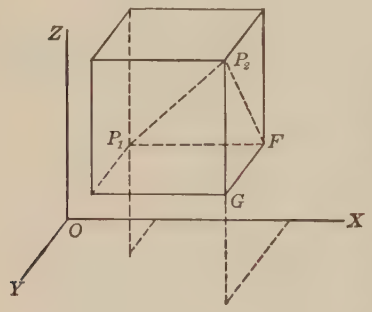


FIG. 169.

PROOF. Through each of the points P_1 and P_2 pass three planes parallel, respectively, to each of the coordinate planes, forming a rectangular solid whose diagonal is P_1P_2, and whose edges P_1F, FG and GP_2 are parallel to the x-, y-, and z-axes, respectively. Then

$$(2) \qquad \overline{P_1P_2}^2 = \overline{P_1F}^2 + \overline{FG}^2 + \overline{GP_2}^2.$$

But $P_1F = x_2 - x_1$, $FG = y_2 - y_1$, and $GP_2 = z_2 - z_1$. Substituting these values in (2), we obtain

$$(3) \qquad P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

which is the required formula.

COROLLARY. The distance from the origin to any point $P(x, y, z)$ is given by the formula

$$OP = \sqrt{x^2 + y^2 + z^2}. \tag{4}$$

282. Point Dividing a Line in a Given Ratio. If the point $P(x, y, z)$ divides the line joining $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ in the ratio

$$\frac{P_1P}{PP_2} = \frac{l_1}{l_2}, \tag{5}$$

then

$$x = \frac{l_1x_2 + l_2x_1}{l_1 + l_2}, \quad y = \frac{l_1y_2 + l_2y_1}{l_1 + l_2}, \quad z = \frac{l_1z_2 + l_2z_1}{l_1 + l_2}. \tag{6}$$

The proof is left to the student.

HINT.

$$\frac{x - x_1}{x_2 - x} = \frac{l_1}{l_2}.$$

EXERCISES

1. Find the length of a line joining

(*a*) $P_1(1, 2, 3)$ to $P_2(6, 8, 10)$.
(*b*) $P_1(4, 7, 3)$ to $P_2(10, 9, 12)$.
(*c*) $P_1(3, -2, 5)$ to $P_2(7, 8, 11)$.

2. Find the coordinates of the mid-point of the line joining (2, 1, 3) and (4, 7, 10).

3. Find the coordinates of the mid-point of each of the lines in Ex. 1.

4. Find the coordinates of the point which divides the line joining (2, 3, 1) and (10, 11, 13) in the ratio of 1 to 3.

5. One extremity of a line is at (1, 4, 2) and the middle point is at (3, 7, 5). What are the coordinates of the other extremity?

6. In what ratio does the point (2, 1, 6) divide the line joining $(-2, -3, 4)$ and (10, 9, 10)?

7. Show that the points $(-3, 2, -7)$, $(2, 2, -3)$ and $(-3, 6, -2)$ are the vertices of an isosceles triangle.

8. Show that the points $(4, 3, -4)$, $(-2, 3, 2)$ and $(-2, 9, -4)$ are the vertices of an equilateral triangle.

9. Find the projections on the axes of the line drawn from $P_1(2, 1, 4)$ to $P_2(6, 9, 7)$.

10. Find the length of a line whose projections on the axes are, respectively,

(*a*) 6, 4, and 2. (*c*) 3, 7, and 5.
(*b*) 2, 8, and 4. (*d*) 3, 7, and -6.

11. Find the length of a line whose projections on the axes are, respectively:

(*a*) 3, 6, and 2. (*c*) -1, 2, and 2.
(*b*) 4, -12, and 3. (*d*) -6, -8, and 24.

12. Find the lengths of the medians of the triangle whose vertices are (1, 2, 4), (4, 6, 10) and (8, 4, 7).

283. Direction Angle. The line OP from the origin to any point P is called the ***radius vector*** and is represented by ρ. The angles which OP makes with the positive directions of the x-, y-, and z-axes are called the ***direction angles*** of the line OP and are denoted by α, β and γ, respectively.

NOTE. The direction angles of a line not passing through the origin are defined as the direction angles of a parallel line passing through the origin and having the same positive direction. The angles α, β and γ are always taken positive and never greater than 180°. They correspond to one direction of the line. The opposite direction corresponds to $180° - \alpha$, $180° - \beta$ and $180° - \gamma$.

284. Direction Cosines. The cosines of the angles α, β and γ are called the ***direction cosines*** of the line OP. They are connected by a very important relation, namely:

The sum of the squares of the direction cosines of any line is equal to 1.

In Fig. 170, let OP be denoted by ρ; then

$$x^2 + y^2 + z^2 = \rho^2. \tag{7}$$

Also

$$x = \rho \cos \alpha; \qquad y = \rho \cos \beta; \qquad z = \rho \cos \gamma. \tag{8}$$

Squaring equations (8) and substituting in (7), we obtain

$$(9)\quad \rho^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = \rho^2.$$

Dividing by ρ^2, we have

$$(10)\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

The direction cosines of the line joining the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given by the equations

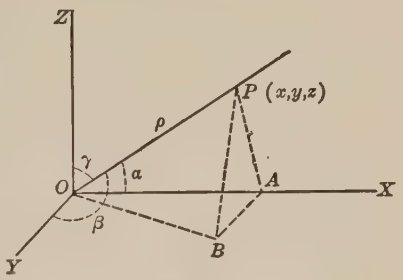


FIG. 170.

$$(11)\quad \cos\alpha = \frac{x_2 - x_1}{d},\quad \cos\beta = \frac{y_2 - y_1}{d},\quad \cos\gamma = \frac{z_2 - z_1}{d},$$

where, as in § 281,

$$d = P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

This is evident from Fig. 169. In the right triangle P_1FP_2,

$$(12)\quad \cos\alpha = \frac{P_1F}{d} = \frac{x_2 - x_1}{d}.$$

Let the student form the equations for $\cos\beta$ and $\cos\gamma$.

EXERCISES

1. Find the projection on each of the axes of the line drawn from the origin to the point (4, 6, 7).

2. What is the projection on each of the axes of the line joining (2, 1, 4) and (8, 4, 6)? Find the direction cosines of the line.

3. Find the direction cosines of the coordinate axes.

4. Find the direction cosines of the line from the origin to (6, 2, 3).

5. A line makes equal angles with the coordinate axes. Find these angles.

6. A straight line in the xy-plane has a slope equal to 3. Find its direction angles and direction cosines.

7. A line makes an angle of 60° with each of two axes. What angle does it make with the third axis?

8. How many lines through the origin make angles of 60° with the x- and y-axes? What angle do they make with the z-axis?

9. If one direction angle of a line is 135°, another 120°, what is the third?

10. A line makes an angle of 45° with the x-axis, and 60° with the y-axis. What angle does it make with the z-axis?

11. Where does a line lie if $\cos\alpha = 0$? $\cos\beta = 0$? $\cos\gamma = 0$?

12. Show by means of direction cosines that the points (2, 1, 5), (5, 3, 11) and (8, 5, 17) lie on a straight line.

13. Show by means of direction cosines that the points (3, − 2, 7), (6, 4, − 2) and (5, 2, 1) lie on a straight line.

14. Find the angle between each axis and a line drawn from the origin to (*a*) the point (6, 8, 10), (*b*) the point (2, 5, 7).

15. Find the length and the direction cosines of a line drawn from

(*a*) $P_1(2, 5, 3)$ to $P(6, 7, 10)$.
(*b*) $P_1(1, 3, -4)$ to $P(5, 8, 3)$.

16. Given $\cos\alpha = \frac{1}{4}$, $\cos\beta = \frac{1}{2}$. Find $\cos\gamma$.

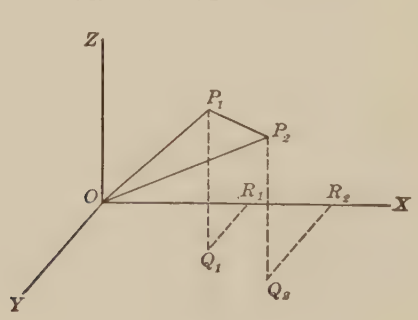


FIG. 171.

285. Angle between Two Lines. Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ represent the direction angles of two given lines, and let OP_1 and OP_2 be two lines through the origin parallel to the given lines, respectively, and θ the angle between them.

Then, if we let $OP_1 = \rho_1$, $OP_2 = \rho_2$ and $P_1P_2 = d$, we have, from the law of cosines,

$$\cos\theta = \frac{\rho_1^2 + \rho_2^2 - d^2}{2\rho_1\rho_2}. \tag{13}$$

But

$$\rho_1^2 = x_1^2 + y_1^2 + z_1^2, \qquad \rho_2^2 = x_2^2 + y_2^2 + z_2^2, \tag{14}$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2. \tag{15}$$

Substituting, we find

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\rho_1\rho_2}. \tag{16}$$

But from (8), § 284, $x_1 = \rho_1 \cos\alpha_1$, $x_2 = \rho_2 \cos\alpha_2$, etc. Hence

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \tag{17}$$

If the lines are perpendicular, $\cos\theta = 0$. Hence the condition for perpendicularity is

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0. \tag{18}$$

If the lines are parallel, they must make the same angles with the axes. Hence the conditions of parallelism are

$$\alpha_1 = \alpha_2, \qquad \beta_1 = \beta_2, \qquad \gamma_1 = \gamma_2. \tag{19}$$

EXERCISES

1. Find the angle between the lines whose direction cosines are, respectively, (6/7, 3/7, − 2/7) and (3/7, − 2/7, 6/7). *Ans.* $\pi/2$.

2. Find the angle between the lines whose direction cosines are (2/3, 2/3, 1/3) and (2/7, 6/7, − 3/7).

3. Show that the points (1, 0, 6), (4, 5, − 2) and (7, 3, 4) are the vertices of a right triangle.

4. Show that the points (4, − 4, 2), (9, − 1, 10), (6, − 7, 8) and (7, 2, 4), taken in that order, are the vertices of a parallelogram.

5. Show that the lines whose direction cosines are (2/3, 1/3, 2/3); (− 1/3, − 2/3, 2/3); and (2/3, − 2/3, − 1/3) are mutually perpendicular.

6. Show that the lines joining the following pairs of points are either parallel or perpendicular.

(3, 2, 7), (1, 4, 6) and (7, − 5, 9), (5, − 3, 8).

286. Loci in Space. If any point P which lies on a given surface be represented by the coordinates (x, y, z), then the condition which defines the surface will lead to an equation involving x, y and z. One equation involving the coordinates of a point in space represents a surface, but its real locus may be trivial. Thus $x = 2$ represents a plane parallel to the yz-plane and two units to the right of it. The equation $x^2 + y^2 + z^2 = 9$ represents the surface of a sphere of radius 3, and center at the origin. The only real values for which $y^2 + z^2 = 0$ are the points for which $y = 0$ and $z = 0$, that is, the points on the x-axis. Hence the real locus is a straight line. Similarly the only real values which satisfy the equation $x^2 + y^2 + z^2 = 0$ are $x = 0$, $y = 0$, and $z = 0$. Hence the locus is a point, the origin.

The locus of a point which satisfies two given conditions is in general a curve, since it is the intersection of two surfaces. Its real points may, however, be finite in number.

Thus the locus of points which satisfy $x^2 + y^2 + z^2 = 16$ and $x = 1$ is the circle in which the plane intersects the surface of the sphere.

287. The Equation of the Sphere. Let $P(x, y, z)$ be any point on the surface of a sphere having the center $C(a, b, c)$ and radius r. Then $CP = r$.

But by the distance formula

$$\overline{CP}^2 = (x - a)^2 + (y - b)^2 + (z - c)^2. \tag{20}$$

Hence

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \tag{21}$$

is the required equation. This equation may be written in the form

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0, \tag{22}$$

where $d = a^2 + b^2 + c^2 - r^2$. Every equation of the form (22) can be reduced to the form (21) by the process of completing the squares.

EXAMPLE. Reduce the equation $x^2 + y^2 + z^2 - 6x - 4y - 8z = 20$ to the form (21).

Completing the squares, $(x - 3)^2 + (y - 2)^2 + (z - 4)^2 = 49$. Hence the center is at (3, 2, 4) and the radius is 7.

As in the case of the circle, the equation represents a point if $r = 0$, or an imaginary sphere if r^2 is negative. If the center of the sphere is the origin (0, 0, 0), the equation of the sphere is $x^2 + y^2 + z^2 = r^2$.

288. Equation of a Cylinder with Elements Parallel to One of the Coordinate Axes. We have already seen that the equation $x^2 + y^2 = r^2$ represents a circle in the xy-plane having its center at the origin and its radius equal to r. Let us now consider a right circular cylinder having the z-axis for its axis and with radius r. If any point P be taken on the surface of this cylinder, the x and y of this point are the same as the x and y of the projection M of this point on the xy-plane. But the latter values satisfy the equation $x^2 + y^2 = r^2$. Hence the equation of the surface of a right circular cylinder having the z-axis as its axis is

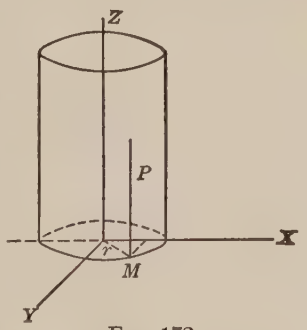


FIG. 172.

$$x^2 + y^2 = r^2. \tag{22a}$$

Likewise the equation of a right circular cylinder having the x-axis for its axis is

$$y^2 + z^2 = r^2. \tag{23}$$

The equation $3x^2 + 2y^2 = 10$ represents a cylinder whose elements are parallel to the z-axis and intersect the ellipse $3x^2 + 2y^2 = 10$ in the xy-plane. Similarly, $y^2 - z = 4$ is the equation of a parabolic cylinder along OX. In general an equation containing only two coordinates represents a cylindrical surface, the elements of which are parallel to the axis corresponding to the missing coordinate.

NOTE. The term *cylinder* is used to denote *any* surface generated by a straight line moving parallel to a fixed straight line.

289. Traces of a Surface. The curve in which a surface intersects a coordinate plane is called the trace of the surface in that plane.

Since $x = 0$ for all points in the yz-plane, and for no other points, the equation of the yz-trace of a surface is found by letting $x = 0$ in the equation of the surface. Similarly, the xy-trace of a surface is found by letting $z = 0$, and the equation of the xz-trace by letting $y = 0$.

The general appearance of a surface may be determined by considering the curves in which it is cut by a system of planes taken parallel to each of the coordinate planes in turn. This also enables one to determine whether the surface is closed or recedes to infinity.

EXAMPLE. The trace of the surface $x^2 + y^2 + z^2 = 16$ on the xy-plane is $x^2 + y^2 = 16$, on the yz-plane $y^2 + z^2 = 16$.

EXERCISES

1. Find the equation of the locus of a point which is

(*a*) 5 units to the right of the yz-plane; 4 units to the left.
(*b*) 4 units above the xy-plane; 7 units below.
(*c*) 3 units in front of the xz-plane; 5 units back.

2. Find the equation of a plane parallel to

(*a*) the xy-plane and 10 units above it; 2 units below it.
(*b*) the xz-plane and 3 units in front of it; 7 units back.
(*c*) the yz-plane and 2 units to the right of it; 4 units to the left.

3. What are the equations of the coordinate planes?

4. What is the form of the equation of a plane which is parallel to the xy-plane? xz-plane? yz-plane?

5. Write the equations of the faces of a rectangular solid which has one vertex at the origin, three edges lying on the coordinate axes, and one vertex at (5, 3, 8).

6. Find the equation of the locus of a point whose distance from the point

(*a*) (0, 0, 0) is 7.
(*b*) (2, 3, 5) is 4.
(*c*) (− 2, 1, 6) is 8.
(*d*) (− 3, − 2, 1) is 5.
(*e*) (− 4, − 5, − 6) is 3.
(*f*) (4, 1, 5) is 6.

7. Find the equation of a sphere having

(*a*) center at (2, − 3, 5) and radius 10.
(*b*) center at (4, − 5, 6) and radius 8.
(*c*) center at (3, − 4, 8) and radius 5.
(*d*) center at (0, 0, a) and passing through the origin.

8. Find the equation of a sphere having

(*a*) the line joining (2, 4, 5) and (6, 10, 7) for a diameter.
(*b*) its center at (3, 2, 5) and its surface passing through (5, 8, 10).
(*c*) its center at (2, 4, 5) and being tangent to the yz-plane.
(*d*) its center at (6, 4, 5) and passing through the origin.

9. Find the equation of a sphere which passes through the points (− 2, 0, 0), (0, − 4, 0), (0, 0, 4) and (8, 0, 0).

10. Describe each of the following surfaces.

(*a*) $x^2 + y^2 = 9$.
(*b*) $y^2 + z^2 = 16$.
(*c*) $x^2 + z^2 = 18$.
(*d*) $2x^2 + 5y^2 = 10$.
(*e*) $y^2 = 2x$.
(*f*) $z^2 = 4y$.

11. Describe each of the following surfaces.

(*a*) $(x - 3)^2 + y^2 = 9$.
(*b*) $(z - 1)^2 + x^2 = 4$.
(*c*) $y^2 - 4y + x^2 = 0$.
(*d*) $\left(\frac{x - 2}{9}\right)^2 + \left(\frac{y - 3}{4}\right)^2 = 1$.
(*e*) $y^2 = 4(x + 1)$.

12. Find the center and radius of each of the following spheres.

(*a*) $x^2 + y^2 + z^2 - 4x - 6y - 2z = 22$.
(*b*) $4x^2 + 4y^2 + 4z^2 - 8x - 24y - 16z = 35$.

13. A point moves so that the sum of the squares of its distances from $(-1, 1, 0)$ and $(0, 0, 1)$ is 6. Prove that its locus is a sphere. Find its center and radius.

14. What conditions are satisfied by all points on

(*a*) the x-axis? (*b*) the y-axis? (*c*) the z-axis?

15. Find the equation of the right circular cylinder

(*a*) whose axis is the z-axis and whose radius is 3.
(*b*) whose axis is the x-axis and whose radius is 5.
(*c*) whose axis is the y-axis and whose radius is 2.

290. Paraboloid of Revolution. A surface formed by the revolution of a plane curve about a fixed line in its plane is called a *surface of revolution*. If the equation of the plane curve is known, the equation of the surface can be obtained from it.

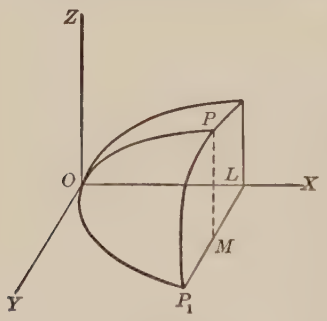


FIG. 173.

Consider the surface formed by revolving the parabola $y^2 = 4x$ about the x-axis. If $P(x, y, z)$ is any point on this surface, then $x = OL$, $y = LM$, $z = MP$.

Since $LP = LP_1$, it follows from the equation of the parabola that $\overline{LP}^2 = 4OL$. But

$$\overline{LP}^2 = \overline{LM}^2 + \overline{MP}^2 = y^2 + z^2. \tag{24}$$

Hence

$$y^2 + z^2 = 4x, \tag{25}$$

the equation of a paraboloid of revolution which passes

through the origin and has the positive half of the x-axis for its principal axis.

291. The Sphere. The circle $x^2 + y^2 = r^2$ may be rotated about the x-axis forming a sphere of radius r. Let P_1 be any point on the circle; then

$$\overline{OL}^2 + \overline{LP}_1^2 = r^2.$$

As the circle rotates about OX, OL remains fixed and equal to x. But LP_1 takes the position LP. Hence

$$\overline{LP}_1^2 = \overline{LM}^2 + \overline{MP}^2.$$

Substituting,

$$\overline{OL}^2 + \overline{LM}^2 + \overline{MP}^2 = r^2. \tag{26}$$

If (x, y, z) are the coordinates of P, $OL = x$, $LM = y$, $MP = z$. Substituting these values, we obtain

$$x^2 + y^2 + z^2 = r^2, \tag{27}$$

which is the equation of the required spherical surface.

292. Surfaces of Revolution. To find the equation of the surface generated by revolving a plane curve about one of the coordinate axes in its plane, substitute in the equation of the curve the square root of the sum of the squares of the two variables which measure distances at right angles to the axis of revolution for that one of these variables which occurs in the equation of the curve.

(a) The equation of the surface generated by revolving the line $y = mx$ about the x-axis is

$$\sqrt{y^2 + z^2} = mx, \quad \text{or} \quad y^2 + z^2 - m^2x^2 = 0, \tag{28}$$

which is therefore the equation of a ***right circular cone*** with its vertex at the origin.

(*b*) The equation of the surface generated by revolving the line $y = mx + c$ about the x-axis is

$$\sqrt{y^2 + z^2} = mx + c, \qquad \text{or} \qquad y^2 + z^2 = (mx + c)^2, \tag{29}$$

which is therefore the equation of a cone.

(*c*) The equation of the surface generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the x-axis is

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1; \tag{30}$$

this surface is called an ***ellipsoid of revolution.***

EXERCISES

1. What is the trace of the surface $x^2 + y^2 + z^2 = 25$ on

(*a*) the xy-plane? (*b*) the yz-plane? (*c*) the xz-plane?

2. What is the trace of the surface of $y^2 + z^2 = 6x$ on

(*a*) the xy-plane? (*b*) the xz-plane?

3. Discuss the locus of each of the following equations.

(*a*) $x^2 + z^2 = 4y$. (*d*) $\frac{x^2}{16} + \frac{y^2 + z^2}{4} = 1$. (*g*) $x^2 + y^2 = 4z^2$.

(*b*) $x^2 + y^2 = 4z$. (*e*) $x^2 + y^2 + 4z^2 = 16$. (*h*) $x^2 + y^2 + z^2 = 0$.

(*c*) $\frac{x^2}{16} + \frac{y^2 + z^2}{9} = 1$. (*f*) $y^2 + z^2 = 9x^2$. (*i*) $x^2 + y^2 + z^2 = 64$.

4. Find the equations of the surfaces of revolution generated by revolving each of the following curves about the axis indicated.

(*a*) $x^2 + y^2 = 25$, x-axis. *Ans.* $x^2 + y^2 + z^2 = 25$.

(*b*) $y^2 = 2x$, x-axis. (*e*) $x^2 + y^2 = 36$, x-axis.

(*c*) $4x^2 + 9y^2 = 36$, x-axis. (*f*) $y = 3x$, x-axis.

(*d*) $4x^2 + 9y^2 = 36$, y-axis. (*g*) $y = 2x + 1$, x-axis.

5. Find the equation of the surface of revolution formed by revolving about the x-axis

(*a*) the line $y = 3$.

(*b*) the line $y = x$.

(*c*) the parabola $y^2 = 2mx$.

293. Definite Integrals. The student has seen that by the process called integration it is possible to find velocities, distances, areas, etc. This process is based upon what is known as the Fundamental Theorem of Integral Calculus, namely: Under proper conditions the limiting value of a sum of terms of the form $f(x)dx$, as the number of terms increases indefinitely, is equal to the definite integral of $f(x)dx$, taken between proper limits.

The solution of each problem requires two distinct steps: (1) setting up the integral and determining the limits of integration; (2) the evaluation of the integral. A careful study of the illustrative examples will be valuable to the student. In § 164 it was shown that the area under a curve $y = f(x)$ from $x = a$ to $x = b$ is given by the definite integral

$$\int_a^b y dx = \int_a^b f(x)dx.$$

A method somewhat similar will now be developed for finding the volume of a solid.

In Fig. 174 the solid described is a solid generated by revolving the area under the curve $y = f(x)$ about the x-axis. Consider the volume between a fixed plane perpendicular to OX at A and a moving plane HL which is at a varying distance from the plane through A and parallel to it. Let $x(OC)$ take on a small increment $\Delta x(CD)$. Then V takes on a small increment ΔV, the volume of the disc generated by revolving the plane surface $CDGF$ about OX. But the disc generated

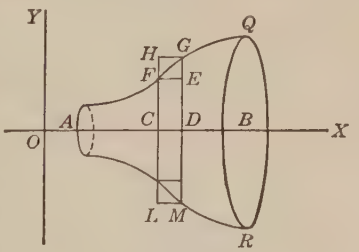


Fig. 174.

by revolving the surface $CDGF$ about OX is less than the cylinder generated by revolving $CDGH$, and greater than the cylinder generated by revolving $CDEF$ about OX. Hence,

$$\pi\overline{CF}^2\Delta x < \Delta V < \pi\overline{DG}^2\Delta x.$$

Dividing by Δx,

$$\pi\overline{CF}^2 < \frac{\Delta V}{\Delta x} < \pi\overline{DG}^2 \tag{31}$$

Now if Δx approaches zero as a limit, DG approaches CF as a limit, and we obtain

$$\frac{dV}{dx} = \pi\overline{CF}^2 = \pi y^2. \tag{32}$$

Using differentials, we may write

$$dV = \pi y^2 dx, \tag{33}$$

which is the differential of the volume of the solid of revolution. Integrating, we have

$$V = \pi \int y^2 dx. \tag{34}$$

Now if $OA = a$ and $OB = b$, the volume generated by revolving the area under the curve $y = f(x)$ about OX from $x = a$ to $x = b$ may be found by means of the formula

$$V = \pi \int_a^b y^2 dx. \tag{35}$$

Similarly, if OY is the axis of revolution, the volume generated between the limits $y = c$ and $y = d$ is given by

$$V = \pi \int_c^d x^2 dy. \tag{36}$$

Example 1. Find the volume of the cone generated by revolving the line $y = x/2$ about OX from $x = 0$ to $x = 6$.

SOLUTION. Here

$$V = \pi \int_0^6 y^2 dx$$

$$= \pi \int_0^6 \frac{1}{4} x^2 dx = \frac{\pi}{4} \int_0^6 x^2 dx = \frac{\pi}{4} \frac{x^3}{3} \Big]_0^6$$

$$= 18\pi \text{ cubic units.}$$

EXAMPLE 2. Find the volume generated by revolving the circle $x^2 + y^2 = 25$ about the x-axis.

SOLUTION. The volume is given by

$$V = 2\pi \int_0^5 y^2 dx$$

$$= 2\pi \int_0^5 (25 - x^2) dx$$

$$= 2\pi \left(25x - \frac{x^3}{3} \right) \Big]_0^5$$

$$= \frac{500\pi}{3} \text{ cubic units}$$

EXERCISES

1. Find the volume of the cone generated by revolving the line $y = x$ about the x-axis from $x = 0$ to $x = 5$.

2. Find the volume of the cone generated by revolving the line $y = 2x$ about the x-axis from $x = 0$ to $x = 3$.

3. Find the volume of the cone generated by revolving the line $y = \frac{1}{4}x$ about the x-axis from $x = 0$ to $x = 4$.

4. Find the volume of the frustum of a cone generated by revolving $y = x$ about the x-axis from $x = 2$ to $x = 6$.

5. Find the volume of the frustum of a cone generated by revolving the line $y = \frac{1}{2}x + 3$ about the x-axis from $x = 0$ to $x = 6$.

6. Find the volume of the paraboloid generated by revolving the curve $y^2 = x$ about the x-axis from $x = 0$ to $x = 4$.

7. Find the volume of the paraboloid generated by revolving the curve $y^2 = 4x$ about the x-axis from $x = 0$ to $x = 4$.

8. Find the volume of the frustum generated by revolving about the x-axis the line $y = x + 3$ from $x = 0$ to $x = 5$.

9. Find the volume of the paraboloid generated by revolving $y^2 = 16x$ about the x-axis from $x = 0$ to $x = 5$.

10. Find the volume of the paraboloid generated by revolving the arc of the parabola $y^2 = 2x$ about the x-axis from $x = 0$ to $x = 5$.

11. Find the volume of the paraboloid generated by revolving the arc of the parabola $y^2 = 8ax$ about OX as an axis from $x = 0$ to the point (x_1, y_1).

12. Find the frustum of the cone generated by revolving the line $y = 2x + 6$ about OX from $x = 0$ to $x = 8$.

13. Find the volume of a sphere generated by revolving the circle $x^2 + y^2 = r^2$ about the x-axis.

14. Find the volume of the hyperboloid of revolution formed by revolving $x^2 - y^2 = a^2$ about the x-axis from $x = a$ to $x = 2a$.

15. Find the volume generated by revolving about the x-axis the part of the curve $y = 9 - x^2$ from $x = 0$ to $x = 3$.

16. Find the volume generated by revolving about the x-axis the curve $y = x^2$ from $x = 1$ to $x = 3$.

17. Find the volume of the ellipsoid generated by revolving $x^2/a^2 + y^2/b^2 = 1$ about the x-axis. Use the limits $x = 0$ and $x = a$.

18. Find the volume of the ellipsoid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the y-axis.

19. Find the volume of the hyperboloid generated by revolving the hyperbola $x^2/a^2 - y^2/b^2 = 1$ about OX as an axis from $x = a$ to $x = 2a$.

20. Find the volume of the spindle-shaped solid generated by revolving the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ about OX as an axis.

21. Find the volume generated by revolving about the x-axis the cubical parabola $y = x^3$ from $x = 0$ to $x = 4$.

294. Work. Work, in a scientific sense, is done only by a force in overcoming resistance, that is, the point of application of the force must move.

(*a*) If a particle moves in a straight line under the action of a *constant* force F, the work done is defined as the product of the force times the component distance passed over in the direction of the force:

$$W = Fx. \tag{37}$$

To compute the work done, multiply the force in pounds by

this distance in feet. The product is the work done expressed in foot-pounds.

EXAMPLE. How much work is done when a force of 25 pounds moves its point of application 8 feet in the direction of the force?

SOLUTION. Since $F = 25$ lbs. and $x = 8$ feet, the work

$$W = 25 \times 8 = 200 \text{ foot-pounds.}$$

(*b*) Consider the *variable* force, whose average value is $\overline{F}$, acting through the distance Δx. If ΔW is the work done, then

$$\Delta W = \overline{F}\Delta x, \quad \text{or} \quad \frac{\Delta W}{\Delta x} = \overline{F}. \tag{38}$$

The instantaneous rate at which W is increasing is, therefore,

$$\frac{dW}{dx} = F. \tag{39}$$

Integrating, we have

$$W = \int F dx. \tag{40}$$

Hence, W is represented by the area under a force-distance graph.

EXAMPLE. The force (F lb.) required to stretch a certain coiled spring a distance of x feet is $F = 12x$. Find the work done in stretching it 2 feet beyond its normal length.

SOLUTION.

$$W = \int F dx = \int_0^2 12x dx = 6x^2 \Big]_0^2 = 24 \text{ foot-pounds.}$$

295. Horse-Power. The horse-power is the unit for the rate of doing work. This unit is 550 foot-pounds per second, or 33 000 foot-pounds per minute. It is equivalent to the work done in lifting a weight of 33 000 lb. through 1 foot, or 1000 lb. through 33 feet, in one minute.

$$\text{Horse-power} = \frac{\text{Work in foot-pounds}}{33\,000 \times \text{time in minutes}}.$$

The value of this unit was adopted by James Watt as the result of experiments with strong dray horses in England. Tests showed that a steady pull equal to one-tenth of the weight of a horse, for a ten-hour daily service at the walking rate of 2.5 miles per hour, was an average of effective service. A 1500-pound horse walking at the rate of $2\frac{1}{2}$ miles per hour and pulling 1/10 of its weight does

$$\frac{150 \times 2\frac{1}{2} \times 5280}{60} = 33\,000 \text{ foot-pounds of work per min.}$$

It is estimated that the average American horse, being somewhat lighter than the English dray-horse, can exert a steady pull of about 120 lb., at the rate of $2\frac{1}{2}$ miles per hour, or do about 4/5 horse-power of work.

It has been estimated that a sprinter may develop as much as nine horse-power at the start of a foot-race.

EXERCISES

1. Find the work done in lifting a weight of 48 lb. to the top of a wall 30 feet high.

2. Find the work done in lifting an automobile weighing 3140 pounds through a vertical height of 18 feet.

3. A horse draws a load having a draft of 140 pounds at the rate of 3 miles per hour. What horse-power is it developing?

4. A man weighing 160 pounds climbs a stairway 36 feet high in 12 seconds. Is he doing a horse-power of work?

5. An elevator lifts a load of 5400 pounds through a distance of 20 feet in 7 seconds. What horse-power is being used?

6. The maximum load on an elevator is limited to 2000 pounds in excess of the weight of the elevator itself, which is 1600 lb. If the elevator fully loaded is lifted through 180 feet in 20 seconds, compute the horse-power required.

7. In an elevator 1000 bushels of oats are lifted through a distance of 22 feet every 8 minutes. Find the rate of doing work.

8. A centrifugal pump discharges 920 cubic feet of water per minute against a total lift of 32 feet. Find the horse-power required.

9. A stream of water has a cross section of 100 square feet. It is flowing at the rate of 4 mi. per hour. At a dam there is a fall of 18 feet. Find the theoretical horse-power.

10. The force (F lb.) required to stretch a given spring x inches is $F = 10x$. Find the work done in stretching it from $x = 0$ to $x = 3$ ft.

11. The force (F lb.) exerted by a given piston is $F = 25x^2$, where x is the distance in feet from one end of the cylinder. Find the work done as the piston moves from the position $x = 2$ ft. to 3 ft.

12. The force (F lb.) used in driving a certain piston is given by $F = 600/x^2$. Find the work done in driving the piston from $x = 5$ to $x = 10$ ft.

13. For a body outside the earth the force (F lb.) with which the earth attracts a "mass of 1 lb." at a distance x miles from the earth's center is $F = (4000)^2/x^2$. Find

(*a*) the force by which a 1 lb. mass at the distance of the moon, 240 000 miles away, is attracted by the earth.

(*b*) the work done in raising a 1 lb. mass from the surface of the earth ($r = 4000$ mi.) to a height of 1000 miles. To a distance of 240 000 miles.

14. The force exerted in expanding a given balloon is $F = 3x^2$ inch-pounds, where x is the radius in inches. Find the work done in increasing the radius from 3 inches to 5 inches.

15. The force in pounds applied to a car is $F = 50x^2 - x^3$. Find the work done between $x = 0$ and $x = 10$ feet.

CHAPTER XVII

PERMUTATIONS AND COMBINATIONS

296. Introduction. A person wishes to go from A to C by the way of B. There are two roads from A to B and five roads from B to C. By how many different routes can he go from A through B to C?

Either road from A to B can be followed by any one of the five roads from B to C. Hence there are 2×5 or 10 routes by which he can go from A to C. This example illustrates the following principle.

297. Fundamental Principle. If one thing can be done in a different ways, and if, after it is done, in any one of these ways, another thing can be done in b different ways, where b is fixed, then the two together can be done in ab different ways. Corresponding to the first way of doing the first thing there are b ways of doing the second thing; corresponding to the second way of doing the first thing there are b ways of doing the second thing; and so on for each of the a ways of doing the first thing. Hence, there are a times b ways of doing the two things together. The following theorem is a generalization of this principle.

THEOREM. *If one thing can be done in a ways, a second thing in b independent ways, a third thing in c independent ways, the three things in this order can be done in abc different ways.*

EXERCISES

1. With three spices and four flavoring extracts how many different tasting drinks can be made, using one of each?

2. There are five trails leading to the top of Mt. Moosilauke, N. H. In how many ways can one go to the top, and return by a different trail?

3. There are 4 routes from A to B. In how many ways can one make the round trip?

4. A hostess who has invited 20 guests wishes to play a game in which each guest has a different number. She finds that she has an abundance of the numerals 1, 2, 3, and 4. Using these numerals one at a time or two at a time, can she have a different number for each guest? Write the numbers.

5. A boy scout has two flags, one red and one white. If each flag can be held in any one of five positions and each different position of the two flags (except together) represents a letter, how many different letters can be represented, using both flags at a time?

6. A committee of three is chosen from 3 physicians, 5 lawyers, and 10 business men so that each group is represented. How many different committees can be chosen?

7. If a coin is tossed, in how many ways can it fall?

8. If a die is tossed, in how many ways can it fall?

9. If two dice are tossed, in how many ways can they fall?

10. If one coin and one die are thrown, in how many ways can they fall?

11. How many straight lines are determined by 10 points on a given circle? by 8 points? by 3 points?

298. Permutations. In certain problems it is necessary to know the number of different ways in which n things can be arranged. This information can be obtained by counting, but the process may be long. The result in a typical problem can be embodied in a formula and numerical results obtained by proper substitution.

DEFINITION. Each different serial arrangement that can be made of a given number of things from a given collection is called a ***permutation.*** The different permutations of the letters a, b, c taken two at a time are ab, ac, ba, bc, ca, cb. Their number is $3 \times 2 = 6$. If they are taken three at a time, the permutations are abc, acb, bac, bca, cab, cba, and their number is $3 \times 2 \times 1 = 6$. The permutations of the letters a, b, c, d taken three at a time are the 24 following.

abc	*bac*	*cab*	*dab*
acb	*bca*	*cba*	*dba*
abd	*bad*	*cad*	*dac*
adb	*bda*	*cda*	*dca*
acd	*bcd*	*cdb*	*dbc*
adc	*bdc*	*cbd*	*dcb*

The special cases just considered lead to a general formula for the number of permutations of n things taken r at a time. This number is represented by the symbol ${}_nP_r$. The first place in the arrangement can be filled by any one of the n things. When this place has been filled, any one of the remaining $n - 1$ things may be chosen for the second place. When these two places are filled, any one of the remaining $n - 2$ things may be chosen for the third place, and so on. Hence two things may be chosen in $n(n - 1)$ ways; three things in $n(n - 1)(n - 2)$ ways. When all but the rth place have been filled we have the choice of any one of the remaining $n - (r - 1)$ or $n - r + 1$ things for this place. Hence the number of permutations of n things taken r at a time is

$$(1) \qquad {}_nP_r = n(n - 1)(n - 2) \cdots (n - r + 1).$$

The student should observe that the first factor in this formula is the total number of things considered. The number of factors is the number of things taken in each arrangement. When $r = n$, formula (1) becomes

$$(2) \qquad P = n(n - 1)(n - 2) \cdots 2 \cdot 1 = n!.$$

STATEMENT. *The number of permutations of n things taken n at a time is factorial n, that is, the product of all the integers from 1 to n inclusive.*

299. Permutations of n Things not All Different. The number of arrangements of the five letters of the word

"place" is 5! or 120. The number of distinct arrangements of the six letters of the word "chance" is not 6! as the six letters of the word are not all distinct. A rearrangement of the two c's would not make a distinguishable difference. Since the number of distinct arrangements of the two letters if they were different is 2, the number of distinguishable arrangements in "chance" is $6!/2 = 360$. This principle is expressed in the following theorem.

THEOREM. *If n things are taken all at a time, in which n_1 are alike, n_2 others alike, n_3 others alike, and so on, then the number of permutations is*

$$P = \frac{n!}{n_1!n_2!n_3!\cdots}. \tag{3}$$

EXERCISES

1. How many different numbers can be written with the digits 1, 2, 3, 4, taken one at a time? Taken two at a time, none repeated? Three at a time, none repeated?

2. How many different numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (repetitions not allowed), if the number is formed of two digits? Of three digits? Of four digits?

3. In a foot ball game, how many different signals can be given with the numbers 1, 2, 3, 4, using one, two, three, or four at a time?

4. How many kinds of single trip local passenger tickets, good only one way, will a railway company need for use on a division that has 6 stations?

5. How many code "words" are possible, each using any two different letters of the alphabet? Any three? Any four?

6. How many signals can be made with 3 flags, red, white, and blue, if the flags are placed in line one above the other?

7. Considering any arrangement of one or more letters a word, how large a vocabulary could be formed with 6 letters, using any number at a time? With 8 letters, using any number at a time? The vocabulary of the average man is said to be less than 5000 words. Webster's unabridged dictionary has approximately 100 000 words.

8. In a contest, a prize is to be awarded to the person who can get the most arrangements from the letters of the word "Canada." How many arrangements are possible, using all the letters each time.

9. How many dominoes are there in a set from double blank to double six?

10. How many numbers can be written with the digits 1, 2, 3, 4, 5, using any number at a time, no digit being repeated?

11. Would 3 dots, 2 dashes, and 1 pause be enough telegraphic symbols for the letters of the English alphabet, the numerals, and four punctuation marks?

300. Combinations. A group of things or elements without reference to the order of the individuals in the group is called a ***combination***. The combinations of a, b, c taken two at a time are ab, ac, bc. The selections ab, and ba, are different permutations but the same combination. If the letters a, b, c are taken three at a time, there are six permutations, namely, abc, acb, bac, bca, cab, cba, but only one combination.

If ${}_nC_r$ denotes the number of combinations of n things taken r at a time, then a formula can be derived giving the relation between ${}_nC_r$ and ${}_nP_r$. Take any combination of r things. With this, $r!$ permutations can be made. Take a second combination of r things. With this, $r!$ permutations can be made. It is easily seen that there are $r!$ permutations for each combination, or

$$ {}_nC_r \times r! = {}_nP_r. \tag{4}$$

Whence,

$$ {}_nC_r = \frac{{}_nP_r}{r!}. \tag{5}$$

Since

$$ {}_nP_r = n(n-1)\cdots(n-r+1), \tag{6}$$

we may write

$$ {}_nC_r = \frac{n(n-1)\cdots(n-r+1)}{r!}. \tag{7}$$

Multiplying both numerator and denominator by $(n-r)!$, we get

$$(8) \qquad {}_nC_r = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!}.$$

EXERCISES

1. How many different committees of 3 persons can be selected from a group of 12 persons?

2. How many straight lines can be drawn through pairs of points selected from 10 points no three of which are in the same straight line?

3. At a meeting of 8 men, each shakes hands with all the others. How many handshakes are there?

4. How many alloys can be made from thirty of the known metals, chosen two at a time, counting one alloy only for each pair of metals? Solve the problem if there are 3 metals in each alloy.

5. A lock contains 6 tumblers (cut pins) each of which can be placed in 20 distinct positions. At a certain arrangement of the tumblers, the lock can be opened. How many different locks can be made so that no two can be opened with the same key?

6. How many parallelograms are formed when a set of 8 parallel lines is crossed by another set of 5 parallel lines?

7. In how many different ways can 10 beads, all of different colors, be strung so as to form a bracelet?

8. In a building containing 6 offices, how many speaking tubes are needed to connect each office privately with every other one?

9. How many combinations consisting of four notes of equal duration can be formed from the eight tones of the major scale? From the thirteen tones of the chromatic scale?

10. How many different sums can be formed with a cent, a nickel, a dime, a quarter, a half-dollar, and a dollar?

11. A set of weights consists of 1, 2, 4, 8, and 16 pound weights. How many different amounts can be weighed?

12. How many different recitation schedules could a student get in registering for 2 out of 10 electives which come at different hours?

13. Chickens have two webs on each foot. They are sometimes marked by punching a hole in one or more of the webs. How many different combinations of distinguishing marks can be made?

14. The combination of a safe consists of figures and letters arranged on three discs. One bears the numbers 0 to 9 inclusive, another the letters A to M inclusive, and the third the letters N to Z inclusive. If the safe opens for but one of these combinations, how many different combinations can be used?

301. Probability. "It is a truth very certain, when it is not in our power to determine what is true, we ought to determine what is most probable."—René Descartes.

The theory of the probability of an event is fundamental in many lines of scientific work, such as statistical studies of physics, biological and social phenomena, theory of errors of measurements, etc. Problems concerning games of chance afford simple illustrations of the theory, but important applications are found in life and fire insurance, the Mendelian Law, etc. The basic ideas underlying these studies are familiar as the following illustrations will show. If a cube whose faces are lettered a, b, c, d, e, and f is thrown, the total number of ways the cube can fall is six, all of which we will assume are equally likely to occur. In a great number of trials the total number of times each face of the cube will turn up is approximately one-sixth of the number of trials.

302. Definition. The *probability* of an event is the ratio of the number of ways in which it can happen (all regarded as equally likely) to the total number of ways in which it can happen or fail.

If a letter is chosen at random from the alphabet, the chance or probability that it will be a is $1/26$. The chance that it will be a vowel is $5/26$, that it will be a consonant is $21/26$, counting only a, e, i, o, u as vowels.

If a is the number of ways in which an event can happen, and b is the number of ways in which it can fail, the probability that the event will happen is p, where

$$p = \frac{a}{a+b}.$$

The probability that it will fail is

$$q = \frac{b}{a+b}.$$

THEOREM. *If p is the probability that an event will happen and q the probability that it will fail,*

$$p + q = 1,$$

for $p = \frac{a}{a+b}$, *and* $q = \frac{b}{a+b}$; *therefore,*

$$p + q = \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

EXERCISES

1. If a coin is tossed, what is the probability of its falling heads?

2. If two coins are tossed, what is the probability of getting (*a*) two heads, (*b*) two tails, (*c*) one head and one tail?

3. If three coins are tossed, what is the probability of getting (*a*) three heads, (*b*) three tails, (*c*) two heads and one tail?

4. If a die is thrown, what is the chance of getting a six?

5. If two dice are thrown, what is the probability of getting (*a*) a double six, (*b*) a sum not less than eight?

6. If three dice are thrown, what is the probability of getting (*a*) all ones, (*b*) all alike, (*c*) a sum not less than 15?

7. A weather bureau predicts rain for to-day. Find the probability of rain, if on the average 92 out of every 100 predictions are correct.

8. On 17 out of 20 school days a student has received a grade of 90% or better. What is the probability that his grade will be at least 90% on the 21st day?

9. What is the probability of selecting Thursday when naming a future date at random?

10. The French philosopher D'Alembert said: "There are two possible cases with respect to every future event; one that it will occur, the other that it will not occur. Hence the chance of every event is 1/2 and the definition of probability is meaningless." Is his statement correct?

303. Probability of Human Life. Mortality Table. An important application of the theory of probability relates to the duration of human life. This problem confronts every life insurance company, every employer who seeks data for a pension system, every judge who seeks to determine the value of a life interest in an estate. It is, therefore, fundamental in the theory of pensions, life annuities, inheritance tax laws, etc. Such problems are based on tables, called mortality tables, which show, out of 100 000 persons of a given age, the number of deaths that may be expected to take place during each of the following years. These tables are based upon observation and not upon mathematical considerations. They differ to some extent for different countries, different races, different occupations, and for different periods of time in the same country.

The table used in this book is called the American Experience Table and is based on the records of thirty American insurance companies. It is the basis for most of the life insurance business in this country.

If a person 20 years old asks the question: "What is the probability that I will live to be 50 years of age?", the question may be answered with reasonable accuracy. From the table it is seen that out of 100 000 persons living at the age of 10 years, 92 637 are living at the age of 20, and 69 804 are living at the age of 50. Hence the answer to the question is 69 804/92 637, or about 0.75, that is, the probability is about 75 out of 100.

304. Value of an Expectation. If a person is to receive $1000 provided a certain event happens, and the probability of the event is 4/5, then the value of his expectation is 4/5 of $1000 or $800. The mathematical value of an expectation is the product of the sum of money to be paid

in case a specific event happens and the probability that the event will happen.

An insurance company agrees with a man 35 years of age to pay \$1000 to his estate in case of death within one year. By the mortality table, the probability that a man 35 years of age will die within the year is .00895 Consequently the value of his expectation is $\$1000 \times (.00895)$ or \$8.95 This discussion does not take into account the question of interest on the premium paid. If we assume that the payment, if due, is paid at the end of the year, and that money is worth 4%, the present value of the expectation is \$8.95/1.04 or \$8.61 Thus a life insurance company could afford to insure the lives of 100 men of age 35 for one year for \$861.00 cash, plus the cost of conducting the business and a reasonable profit. Compare this premium with the actual premium charged for this age.

EXERCISES

In the first eight problems find the probability:

1. That a cubical die will fall with the number 5 up.

2. That a boy aged 10 will live to graduate from high school at the age of 19.

3. That a coin will fall head up.

4. That a boy aged 16 will live to be 60 years old.

5. That out of a class of 22 boys and 10 girls, if one is selected at random, the one selected will be a girl.

6. That a man aged 21 will live to receive an inheritance at the age of 36.

7. That a man aged 30 will live 20 years.

8. That a man 24 years of age will live to be (*a*) 30 years old; (*b*) 40 years old; (*c*) 60 years old; (*d*) 80 years old.

9. What is the probability that a man now 25 years old will be living 10 years hence? 20 years hence? That he will reach the age of 60 years?

10. A boy is 10 years old. What is the probability that he will live long enough to vote? Live to be 50 years old?

11. A young man just 21 years old is to receive an estate of $8000 on his 35th birthday, provided he lives that long. Find the value of his expectation.

12. A person is to receive $8 in case he tosses four coins and they all fall heads up. What is the value of his expectation?

13. Find the net premium for insuring a man's life for one year for $5000, if he has just reached the age of 25 years. The age of 40 years. The age of 60 years.

14. What would be the cost of insuring a house for $3000 (overhead expenses not included) if, on the average, 3 houses out of every 1300 are destroyed during the year?

15. A stake of $24 is made contingent upon getting a sum greater than 10 in a single throw with two dice. Find the mathematical value of the expectation.

16. Of 100 000 children 10 years of age, 749 die within the year. Neglecting interest, cost of administration, etc., find the cost of insuring the life of a 10-year-old child for $1000 for one year.

17. A man 42 years old starts an enterprise requiring 10 years to complete. What is the probability that he will live to see the work completed?

18. At the close of the Civil War in 1865 approximately 2 800 000 men were mustered out of the two armies. What was the probability that a man 22 years old at the time would be alive in 1929?

19. In playing a game for $100, what is the mathematical value of the expectation of a player whose probability of winning is 3/16?

20. If money is worth 5%, find the present value of the expectation of a man aged 30 who is to receive $8000 in 6 years, if he is still alive.

305. Independent Events. Two events are said to be independent when the occurrence of one of them is not affected by the occurrence of the other.

THEOREM. *The probability that two or more independent events will all happen is equal to the product of their respective probabilities.*

Suppose two coins are tossed. The probability that one will come up heads is 1/2 and the probability that the other will come up heads is also 1/2. Therefore, the probability

that both will come up heads is $1/2 \times 1/2$ or $1/4$. Similarly the probability that three coins will come up heads is $1/2 \times 1/2 \times 1/2$ or $1/8$.

306. Proof of Theorem. Suppose the probabilities of two separate events are p_1 and p_2, respectively. Let a_1 be the number of ways in which the event corresponding to p_1 can happen, and b_1 the number of ways it can fail; similarly let a_2 be the number of ways in which the event corresponding to p_2 can happen, and b_2 the number of ways in which this event can fail; then,

$$p_1 = \frac{a_1}{a_1 + b_1}, \qquad p_2 = \frac{a_2}{a_2 + b_2}.$$

We may associate together any one of the $a_1 + b_1$ cases in which the first event happens or fails, and any one of the $a_2 + b_2$ cases in which the second happens or fails; hence there are $(a_1 + b_1)(a_2 + b_2)$ cases, equally likely to occur. In a_1a_2 of these cases both events happen. Therefore, the probability p that both events happen is

$$p = \frac{a_1a_2}{(a_1 + b_1)(a_2 + b_2)}.$$

But from this equation,

$$p = \frac{a_1}{a_1 + b_1} \times \frac{a_2}{a_2 + b_2} = p_1p_2.$$

This proof can be extended to cover any number of independent events.

307. Dependent Events. If from a bag containing 4 white balls and 3 black balls one ball is drawn at random, the chance that it will be white is $4/7$. If this ball is not replaced, the chance of drawing a second white ball is $3/6$. Hence the chance of twice drawing a white ball when the one drawn first is not replaced is $4/7 \times 3/6 = 2/7$. In

general, if the probability of a first event is p_1, and after this event has happened the probability of a second event is p_2, the probability that both events will occur in the order stated is p_1p_2.

EXERCISES

1. The probability that A will live 10 years is 1/6 and the probability that B will live 10 years is 1/10. Find the probability that they will both live 10 years.

2. A man 36 has a son 15. What is the probability that both will be alive when the father is 50?

3. A bridegroom of 25 marries a bride of 20. What is the probability that they will live and celebrate their golden wedding? Assume the probability of divorce to be one out of seven marriages.

4. The chance of a door being locked is 1/4. The key to the door is on a ring containing 5 keys. If I select one key at random, what is the chance of opening the door?

CHAPTER XVIII

THEORY OF MEASUREMENTS

308. Introduction. A common problem in scientific work is to arrange data in such a way that the reader may draw proper conclusions. A life insurance company tabulates the age at death of thousands of persons. From these data it builds a mortality table showing the life expectation of a person who has reached a certain age. The engineer measures the length of an arc of 1° at different points on the earth's surface and tells us the earth's shape and dimensions.

The astronomer measures the distance to the sun and finds it to be 92 900 000 ± 500 000 miles. The financier draws a graph showing the fluctuation in the purchasing power of the dollar. From such a graph he may anticipate prices due to war or during a period of depression.

In this way the theory of measurements is applied in the physical and biological sciences, medical theory, engineering, astronomy, statistical work in education, and in many other sciences.

309. Small Errors. Any attempt to measure the magnitude of a quantity is subject to error, the error being the difference between the real value and the measured value. The ratio of the actual error to the quantity under consideration is called the *relative error*. The relative error when expressed in per cent is called the percentage of error.

For example, the distance to the moon at a given time is found to be 240 000 ± 1200 miles. The thickness of a watch spring is 0.016 ± 0.001 inch. In the first instance the relative error is 1 in 200 and the percentage of error is one-half of 1%; in the latter the relative error is 1 in 16 and the percentage of error is more than 6%.

If Δx is used to denote the error (assumed to be small) in an independent variable x, then the value of the relative error in a dependent quantity y may be expressed as $\Delta y/y$. If the measurement is fairly accurate, Δy approaches dy numerically and the relative error is closely approximated by the numerical value of dy/y.

EXAMPLE. The radius of a sphere is 4 inches, with a possible error of 0.01 inch. Find approximately (*a*) the greatest possible error in the calculated volume, (*b*) the greatest possible relative error in the calculated volume.

SOLUTION. (*a*) We have given $r = 4$, and $dr = 0.01$, and $V = (4/3)\pi r^3$. Differentiating, we find

$$dV = 4\pi r^2 dr.$$

Substituting the values of r and dr, we obtain

$$dV = 4\pi(16)(0.01) = 2.01 \text{ cu. in.},$$

the greatest possible error.

(*b*) The relative error is the ratio of dV to V, i.e.,

$$\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = \frac{3dr}{r} = \frac{3(0.01)}{4} = 0.0075 = \frac{3}{4} \text{ per cent}$$

310. Error in a Sum. Let a and b be two measurements and Δa and Δb be the respective errors. Then the true values are given as $a \pm \Delta a$ and $b \pm \Delta b$. The sum of the magnitudes is

$$a \pm \Delta a + b \pm \Delta b = a + b \pm \Delta a \pm \Delta b.$$

Since $a + b$ is the sum of the measured values, the error cannot be greater than $\Delta a + \Delta b$.

EXAMPLE. If $a = 96 \pm 0.02$ ft. and $b = 114 \pm 0.05$ ft., the error in the sum cannot exceed $0.02 + 0.05 = 0.07$ ft. Hence the true value of the sum is between $210 + 0.07$ and $210 - 0.07$, that is, 210 ± 0.07 ft.

The relative error is not greater that $0.07/210 = 0.0003$; hence the percentage of error is about 0.03%.

311. Error in a Product. Let a and b be two positive measurements and Δa and Δb be the respective errors. Then the product is $ab \pm a\Delta b \pm b\Delta a \pm \Delta a\Delta b$. Since ab is the product of the measurements and Δa and Δb are supposed to be small in comparison with a and b, the error is quite well represented by $a\Delta b + b\Delta a$.

EXAMPLE. If $a = 100 \pm 0.2$ and $b = 200 \pm 0.3$, the greatest error possible in the product is approximately $100(0.3) + 200(0.2) = 70$. Hence, the true value lies between $20\,000 + 70$ and $20\,000 - 70$, approximately.

It will be seen that the product may be less accurate than either of the factors forming it. Only in the case where one error is positive and the other negative will the errors tend to compensate each other.

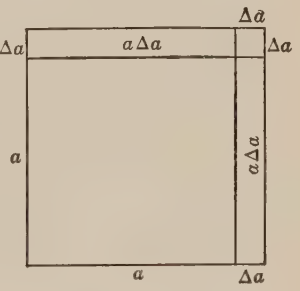


FIG. 175.

312. Error in a Square. Let Δa be the error in reading a (see Fig. 175). Then the area is $(a \pm \Delta a)^2$ and the error is $\pm 2a\Delta a + \overline{\Delta a}^2$. But if Δa is small in comparison with a, the error is quite well represented by $2a\Delta a$.

313. Error in a Square Root. Let Δa be the error in the reading a. Since $a + \Delta a$ is nearly equal to

$$a + \Delta a + \frac{\overline{\Delta a}^2}{4a},$$

$\sqrt{a \pm \Delta a}$ is nearly equal to

$$(1) \qquad \sqrt{a \pm \Delta a + \frac{\overline{\Delta a}^2}{4a}}, \qquad \text{or} \qquad \sqrt{a} \pm \frac{\Delta a}{2\sqrt{a}}.$$

Hence if Δa is the error in the measurement a, the error in the square root of this measurement is approximately $\frac{\Delta a}{2\sqrt{a}}$.

EXAMPLE. If $a = 225 \pm 0.6$, the error in $\sqrt{a}$ is $\dfrac{0.6}{2 \times 15} = \dfrac{0.6}{30} = 0.02$

EXERCISES

Calculate, using the approximate methods just explained, the greatest possible error in the:

1. Sum of $a = 100 \pm 0.04$ and $b = 98 \pm 0.03$ *Ans.* 0.07
2. Product of $a = 60 \pm 0.1$ and $b = 80 \pm 0.2$ *Ans.* 20.
3. Area of a square whose side is $a = 50 \pm 0.1$ *Ans.* 10.
4. Area of a rectangle if $l = 40 \pm 0.2$ and $w = 30 \pm 0.15$ *Ans.* 12.
5. Area of a circle if $r = 10 \pm 0.2$
6. Volume of a cylinder of radius r and height 4 ± 0.01
7. Find the approximate increase (or error) E in the volume of a sphere due to an increase (or error) e in the radius r. *Ans.* $E = 4\pi r^2 e$.
8. What is the approximate increase (or error) E in the volume of a cylinder of known height h, due to an increase (or error) e in the radius r?
9. What is the approximate increase in the volume of a rectangular solid of known base 36 sq. ft., due to an increase (or error) of 0.04 ft. in the height h?
10. The dimensions of a bin, measured to the nearest foot, are as follows: length 8 ft., width 4 ft., height 5 ft. How large may the volume actually be? How small may it be?

SUGGESTION. $l = 8 \pm 0.5$, $w = 4 \pm 0.5$, $h = 5 \pm 0.5$ *Ans.* 210.375; 118.125

11. The floor of a room is found to be 30 by 20, each dimension being to the nearest foot. How should the area be stated? *Ans.* 600 ± 25 sq. ft. or 600 sq. ft.
12. Find the percentage of error if (*a*) the distance to the sun is given as $92\,000\,000 \pm 400\,000$ miles; (*b*) the diameter of the sun is given as $865\,000 \pm 1000$ miles; (*c*) the distance to the star Alpha Centauri is $4 \pm \frac{1}{2}$ light years; (*d*) the diameter of a ball is $1 \pm 1/16$ inches.
13. A line is measured with a 100-ft. steel tape and found to be 1328.4 feet. If the tape is 0.03 foot too long, what is the true value of the length of the line?
14. A certain 100-ft. steel tape which is standard length at 60° F. is found to expand 0.000006 of its length for each degree of rise in tempera-

ture. A line is measured when the temperature of the tape is 97° F. and found to be 1282.6 feet. Find its true length.

15. What is the possible error in the area of a rectangle whose length is 50.4 ft., and width 30.6 feet, due to the measurements being made only to the nearest tenth?

SUGGESTION. Since the measurements are only to the nearest tenth, the length may be 50.4 ± 0.05 and the width 30.6 ± 0.05

16. Show that if Δa is the error in the edge of a cube, then the error in the volume is approximately $3a^2\Delta a$.

17. The radius of the earth's orbit is 92 900 000 miles. What would be the approximate increase in the length of the orbit if the radius were increased to 93 000 000 miles?

18. Suppose an iron hoop girdles the earth at the equator, radius 3960 miles. By how much would the circumference of the hoop have to be increased in order that the hoop may be exactly 6 inches from the surface of the earth at all points? *Ans.* 12π inches.

19. The period of a simple pendulum is $T = 2\pi\sqrt{l/g}$. How much will the period be changed by increasing l from 100 cm. to 100.1 cm.? Take $g = 980$.

SUGGESTION. Find the differential.

$$dT = \frac{2\pi}{\sqrt{980}} \times \frac{dl}{2\sqrt{l}} = \frac{\pi}{\sqrt{980\,l}}\,dl.$$

20. If each edge of a cubical block of metal, when heated, increases 1/10 per cent per degree increase in temperature, show that the surface increases 2/10 per cent per degree.

21. Show that the relative error in the volume of a sphere due to an error in measuring the radius is three times the relative error of the radius.

22. Show that the relative error in the product of two numbers may be as great as the sum of the relative errors of the numbers.

SUGGESTION. Let $y = uv$; then $dy/y = du/u + dv/v$

314. Averages. An ***average*** or ***mean*** is any representative measurement which is used to describe a whole collection of values or objects. Different kinds of averages, their meaning and use will be considered. There are five kinds of averages in common use, namely: the *arithmetic mean*, the

weighted arithmetic mean, the *geometric mean*, the *median*, and the *mode*.

315. Arithmetic Mean or Average. (*a*) *Measurements all different.* When the word mean, average, or arithmetic average is used, it is commonly understood to be the arithmetic mean.

$$\text{(2)}\quad \text{arithmetic mean} = \frac{\text{sum of all the measurements}}{\text{number of measurements}}.$$

If there are n measurements M_1, M_2, $\cdots$, M_n, and A represents the arithmetic mean or average, then

$$\text{(3)}\qquad A = \frac{M_1 + M_2 + M_3 + \cdots + M_n}{n} = \frac{\Sigma M}{n}.$$

EXAMPLE. A boy, using a protractor, measured an angle 5 times. The readings were:

$10^\circ\ 30'\ 0''$; $10^\circ\ 30'\ 10''$; $10^\circ\ 30'\ 20''$; $10^\circ\ 29'\ 50''$; $10^\circ\ 29'\ 55''$.

Find the average of the measurements.

$$\text{average} = \frac{\text{sum of measurements}}{\text{number}} = 10^\circ\ 30'\ 3''$$

(*b*) *Measurements alike.* If there are a large number of measurements, there will frequently be a tendency for them to cluster about some central value. In this way any given value may occur not once, but several times. In such problems if M_1, M_2, $\cdots$, M_n represent the measurements and W_1, W_2, $\cdots$, W_n represent the ***frequencies*** (that is, the number of times the given measurements occur) of the respective measurements, then

$$\text{(4)}\qquad A = \frac{W_1M_1 + W_2M_2 + \cdots + W_nM_n}{W_1 + W_2 + \cdots + W_n}.$$

A is called the ***weighted arithmetic mean*** and is obtained by multiplying each measurement by its frequency and then

dividing the sum of these products by the total number of measurements.

EXAMPLE. Suppose a man gathers 100 ears of corn whose lengths to the nearest inch are given in the table:

Length in inches	8	9	10	11	12	13
Frequency	4	20	40	23	10	3

The weighted arithmetic mean

$$= \frac{4\times8+20\times9+40\times10+23\times11+10\times12+3\times13}{100} = 10.24 \text{ in.}$$

316. Weighted Arithmetic Mean. The weighted arithmetic mean, defined by equation (4), is widely used in determining the average grades as in Civil Service examinations where each subject is weighted, in mixing compounds of different purity, in locating the center of gravity, etc. The various grades of a student, written lessons, recitations, quizzes, final examination, etc., are almost invariably weighted in some way or other.

EXAMPLE. A high school boy carries 15 hours of class work per week and makes the following grades. If the subjects are counted equally hour for hour, what would his average grade be?

COURSE	HOURS PER WEEK	GRADE %	PRODUCT
English	2	88	176
Mathematics	5	96	480
Language	5	80	400
Science	3	78	234
Totals	15		1290

$$\text{weighted arithmetic mean} = \frac{1290}{15} = 86\%.$$

In reporting the average yield per acre, the average height or chest measure of certain classes of men, the mean daily

temperature, or the average wage, one or the other of the above averages is meant.

317. The Geometric Mean. The geometric mean is used in finding the average rate of increase in such problems as growth of population, output of a factory, relative change in prices, etc.

The geometric mean of several terms is defined by the equation

$$G = \sqrt[n]{M_1 M_2 \cdots M_n}. \tag{5}$$

The mean is easily calculated by using logarithms.

EXAMPLE. The population of a town increases from 1000 to 2000 in ten years. What is the average annual rate of increase?

SOLUTION. The arithmetic mean if taken would give 10%. But an annual increase of 10% would give a population of $(1.10)^{10}$ or 2.59 times the population at the beginning of the decade instead of 2. What is wanted is $\sqrt[10]{2} = 1.07 +$, that is, the increase is a little over 7% per year. The population at the end of the successive years is

$$1000\sqrt[10]{2},\ 1000\sqrt[10]{2^2},\ \cdots, \quad \text{or} \quad 1000(1.07 +),\ 1000(1.07 +)^2,\ \cdots,$$

or about 1070, 1145, $\cdots$.

EXERCISES

1. What average is meant by (*a*) the average rainfall; (*b*) the average height of soldiers in a regiment; (*c*) the average rate of increase in population; (*d*) the average rate of increase in prices?

2. In an 8th grade examination, 2 pupils received 60%, 4 received 65%, 8 received 70%, 25 received 80%, 20 received 85%, 12 received 90%, 4 received 95%, and 1 received 100%. What is the average grade for the 76 pupils?

3. A student's semester grade is to be determined as follows: the grade for daily work is to have a weight of 5; the average of tests, a weight of 1; the final examination, a weight of 2. If his grades are (*a*) 84 for daily work, (*b*) 90 for average of tests, (*c*) 72 for final examination, find his semester grade.

4. If land increased from \$50 to \$150 an acre in the decade from 1910 to 1920, find the average (geometric) yearly increase in per cent.

5. The following table shows the increase in the native and foreign-born population of Portland, Oregon, according to the census report. Compare the per cent of increase, decade by decade, from 1870 to 1910.

Year	Native	Foreign	Year	Native	Foreign
1870	5 400	2 600	1900	64 600	25 900
1880	11 300	6 300	1910	163 400	43 800
1890	29 100	17 300			

Represent the above data by means of a graph.

6. A test shows that the protein content of corn was increased 60% in ten years' breeding. What is the average (geometric) annual rate of increase?

7. The average weekly wages of three factories are 16, 22, and 25 dollars per employee, respectively. The number of employees is 290, 375, 492, respectively. Find the average wage for the three factories. What average did you use? Why?

8. The population of the United States was 3 900 000 in 1790 and 105 700 000 in 1920. What was the average percentage rate of increase for each decade of this period, assuming that the population increased in geometric progression with a uniform rate for the entire time?

9. A merchant increased his capital the first year by 15%, the second year by 20%, and the third year by 22%. What was the average annual rate of increase for the three years?

10. A candidate received the following grades in a Civil Service examination. In penmanship 88% weighted 1; in geography 82% weighted 2; in U. S. history 84% weighted 3; in English 94% weighted 3; in arithmetic 96% weighted 4. Compare his standing with the standing of the candidate who has, for the same subjects and weights, the marks of 85% in penmanship; 92% in geography; 86% in history; 90% in English, and 90% in arithmetic.

11. The protein content of corn in a certain community increased year by year, 4%, 6%, 7%, 15%, 22% in five successive years of breeding. What was the average (geometric) rate of increase?

318. Frequency Distributions. Objects may be grouped with respect to some characteristic, such as the weight of ears of corn having a given length, number of students making a given grade, etc. Such a classification showing

the number of individuals in each class is called a ***frequency distribution.*** A graphic representation of such a distribution in which the magnitudes are represented as abscissas, the frequencies as ordinates, and the resulting points connected by straight lines is called a ***frequency polygon.*** If the representation is made by means of rectangles with equal bases, as in Fig. 176, in which the altitudes are equal to the frequencies, and the common length of the base represents the ***class interval,*** the diagram is called a ***histogram.*** The area of any histogram is the sum of the frequencies times the class interval.

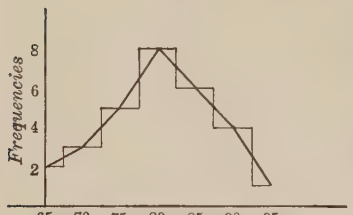


FIG. 176.

The following frequency table shows the number of students making a given grade. The class interval is 5%.

Grade (%)	65	70	75	80	85	90	95
No. of Students	2	3	5	8	6	4	1

319. The Median. The *median* of a group of measurements is the middle measurement if there is one. Thus the median for the group 1, 3, 4, 5, 7, 8, 9 is 5. For the group 2, 6, 8, 9 the median is not defined, but may be taken as 7, which is halfway between the two middle numbers 6 and 8. Thus if all the measurements of a series are numbered in order of magnitude, the magnitude of the measurement which is halfway up the series is called the *median.* If n (the number of measurements) is small, the median may be found conveniently by pairing off the largest and smallest measurements, and repeating the process until only one or

two are left. If only one is left it is the median; if two are left the median is usually taken as the measurement halfway between them. Hence the ordinal number of the measurement whose magnitude is the median is $(n + 1)/2$.

The magnitudes one-quarter and three-quarters up the series are called the first and third ***quartiles,*** respectively, written Q_1 and Q_3. The ordinal numbers of the measurements whose magnitudes are the quartiles are $(n + 1)/4$ and $3(n + 1)/4$, respectively.

EXAMPLE. A factory employs 239 men. If the men are arranged in serial order with respect to wages, the wage of the 120th man is the median wage, the wages of the 60th man and 180th man are Q_1 and Q_3, respectively.

In a frequency table where the measurements are given by classes it is assumed in the computation of the median and quartiles that the measurements are distributed uniformly in each interval so that, after having determined the intervals in which the median and quartiles are to be found, their positions in the respective class intervals are found by ordinary interpolation.

NOTE. The median has the advantage of being easily determined and if the graph approaches the usual distribution it is very near the arithmetic mean. However, since its value is determined by the number of measurements above or below it, extreme values do not affect it and great changes in the data can be made without disturbing the median.

EXAMPLE Find the median and quartiles for the following test.

Grade in %	45	55	65	75	85	95
No. of grades	3	9	22	26	17	3

The graph of this data is shown in Fig. 177.

In this case, the median is not readily distinguishable. Hence the data are rearranged so as to show the total number of grades not greater

than the upper limit in the successive intervals. Since the class interval is 10%, it may be assumed that the grades are distributed uniformly throughout the class intervals of which the above grades are approxi-

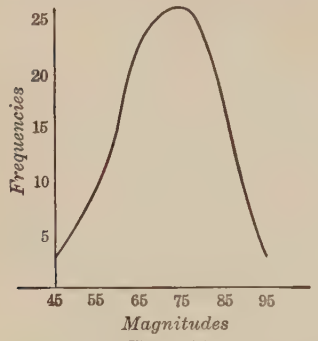


FIG. 177.

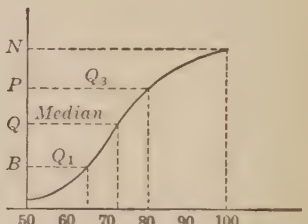


FIG. 178.

mately the mid-points. These intervals may be taken as 41 to 50, 51 to 60, etc.

Grade in % not greater than	50	60	70	80	90	100
No. of grades	3	12	34	60	77	80

The graph is shown in Fig. 178.

COMPUTATION. Dividing the number of grades plus 1 by 2, 81/2 = 40.5 The required median is the abscissa of the ordinate halfway between the 40th and 41st grade. It may be found as follows: From the data it is seen that there are 34 grades of 70% or less, and 60 grades of 80% or less. Hence the required median grade lies somewhere between 70% and 80%. The interval from 34 to 60 is 26; the interval from 34 to 40½ is 6½. Hence, to 70% must be added 6½/26 of 10% or 2½%. 70% + 2½% = 72½%, the median grade. This means that one-half of the grades are below and one-half above 72½%, it being assumed that the grades are evenly distributed.

Q_1. The quartiles are found as follows. Since ¼ of 81 = 20¼, the first quartile grade Q_1 lies between 60% and 70%. The interval from 12 to 34 is 22; the interval from 12 to 20¼ is 8¼. Therefore, to 60% must be added 8¼/22 of 10%, making 64%, nearly.

Q_3. Since $\frac{3}{4}$ of 81 = $60\frac{3}{4}$, the third quartile grade lies between 80% and 90%. The interval from 60 to 77 is 17; the interval from 60 to $60\frac{3}{4}$ is $\frac{3}{4}$. Hence the required grade is

$$80\% + \frac{3/4}{17} \text{ of } 10\% = 80.5\%, \text{ nearly.}$$

320. The Mode. The *mode*, the most common score, "the fashion," is the magnitude of the most frequent item. If in a graph the magnitudes are represented as abscissas, and the frequencies as ordinates, the mode is the abscissa corresponding to the maximum ordinate. In a church offering the following coins were given.

Value of Coin	1¢	5¢	10¢	25¢	50¢	$1	$5
Number of Coins	3	30	14	5	4	2	1

In this illustration the mode is 5¢, since more 5¢ coins were given than any other amount. The mode is sometimes used to characterize a type. Thus by "*the average citizen*" we usually mean the type which is more often met than any other. The mode can be determined, roughly, more easily than either the median or mean, and is not affected by variations at the ends. The disadvantages of the mode lie in the difficulty of determining it accurately. In general the mode is not a desirable average to use in accurate work. It fluctuates too much with slight changes in data near the middle.

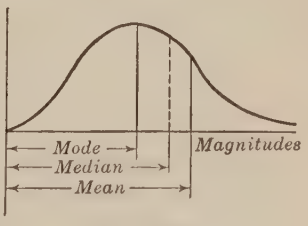


FIG. 179.

In a distribution of the most symmetric type, the arithmetic mean, the median, and the mode fall together. In certain important asymmetrical types, the mode, median, and mean have been observed to fall

in the order named with the mean toward the longer branch of the curve. See Fig. 179.

EXERCISES

1. The following table shows the number of families out of 375 having an income not greater than a given amount. Find the median and quartiles.

Income not greater than ($)	500	600	700	800	900	1000	1100	1200	1300	1400
No. of families	8	25	97	176	249	312	343	361	369	375

2. In a mental test the following grades were given. Find the median and the quartiles.

Grades in %	50–59	60–69	70–79	80–89	90–99	100–109	110–119	120–129
No. of students with that grade	2	5	15	30	50	40	25	2

3. Take the grades given in the last examination in your class and find (*a*) the mean, (*b*) the median and the quartiles, (*c*) the mode.

4. In a memory test given to *A* and *B*, series of 10 letters were read at the rate of 1 per second. As soon as the reading was finished each time, they were required to write in the proper order as many letters as they could remember. Their scores were:

Letters Correct	3	4	5	6	7	8	9	10
A's Frequency	1	3	7	12	15	18	5	3
B's Frequency	0	4	6	15	20	15	4	0

Draw a graph of the performance in each case. Find the median and quartiles. Which is the better performance?

5. The lengths of 1000 ears of corn were measured and the data arranged as follows.

Length in inches.....	4	5	6	7	8	9	10	11	12	13
Number in each group.	6	28	110	164	212	204	180	78	14	4

Find (*a*) the average, (*b*) the median and quartiles, (*c*) the mode.

321. Measure of Variation. Deviation. Next in importance to the average is the deviation, which is the measure of dispersion, or variation of a distribution from the average chosen. If groups are to be compared, two things are essential: (*a*) an *average*, A; (*b*) a *measure of variability*, or *deviation*, $\pm d$. The former obviates the necessity of stating all the measurements from which it was derived. The latter, by a single number, shows the extent to which the other values cluster about or are dispersed from the average chosen. Thus any series of measurements can be summarized by two numbers, $A \pm d$. The principal measures of variability are: (*a*) *the quartile deviation*, Q; (*b*) *the mean deviation*, $A.D$; (*c*) *the median deviation*, or *probable error*, and (*d*) *the standard deviation*, σ.

322. The Quartile Deviation. The quartile deviation is defined by the equation $Q = (Q_3 - Q_1)/2$, where Q_1 and Q_3 are the first and third quartiles, respectively. The quartile deviation for the distribution of grades in the example of § 319 is

$$Q = \frac{80.5\% - 64\%}{2} = 8.2\%.$$

Hence the group may be summarized by the median value of the group and the quartile deviation, namely, $72.5\% \pm 8.2\%$. This means that approximately one-half of the grades lie between $72.5\% + 8.2\% = 80.7\%$ and 72.5%

$-8.2\% = 64.3\%$. The quartile deviation is the simplest measure of deviation to calculate.

323. The Mean or Average Deviation (A.D.). The mean or average deviation is the arithmetic mean of the numerical values of the deviations from the average, each measure being thought of as differing or deviating from the average (either mean or median) of all the measurements.

DATA ARRANGED FOR COMPUTING MEAN DEVIATION AND STANDARD DEVIATION (See § 325)

VALUES	FREQUENCY	DEVIATION		
	f	d	fd	fd^2
15	3	4	12	48
14	7	3	21	63
13	12	2	24	48
12	20	1	20	20
11	25	0	0	0
10	22	1	22	22
9	13	2	26	52
8	7	3	21	63
7	0	4	0	0
	$\Sigma f = 109$		146	$\Sigma = 316$

NOTE. The true mean

$$= \frac{15 \times 3 + 14 \times 7 \cdots + 8 \times 7 + 7 \times 0}{109} = 11.07$$

The approximate mean 11 will be used in this problem. It is probably as accurate as the table warrants.

The mean deviation is the arithmetic mean of the deviations from the average; hence it equals the sum of the entries in the fd column divided by the sum of the frequencies. The mean or average deviation in the above problem is $146/109 = 1.3$ Hence 50% of the measurements may be regarded as lying between the values 11 ± 1.3, that is, between 12.3 and 9.7

324. Median Deviation, or Probable Error. If all the deviations from a given average, mean, or median are

arranged in order of magnitude without regard to sign, the median deviation is the median of these items. It is calculated in the same way as the median of the distribution. The median deviation is usually referred to as the ***probable error.*** See also § 332.

325. The Standard Deviation (S.D. or more often "Sigma"). The standard deviation σ is the square root of the arithmetic mean of the squares of the deviations from the average, and is defined by the equation,

$$(6) \qquad \sigma^2 = \frac{d_1^2 + d_2^2 + \cdots + d_n^2}{n} = \frac{\Sigma d^2}{n},$$

where $d_1, d_2, \cdots, d_n$ are the deviations of the measurements from the average chosen. If the deviations occur with the frequencies $f_1, f_2, \cdots, f_n$, then

$$(7) \qquad \sigma^2 = \frac{f_1 d_1^2 + f_2 d_2^2 + \cdots + f_n d_n^2}{f_1 + f_2 + \cdots + f_n} = \frac{\Sigma f d^2}{\Sigma f}.$$

In general, find the deviation of each measurement from the mean; square each deviation and multiply it by its frequency; add the products, and divide by the total number of measurements and extract the square root. This statement is symbolized by the formula

$$(8) \qquad \sigma = \sqrt{\frac{\Sigma f d^2}{n}}.$$

In the example of § 323, $\sigma^2 = 316/109 = 2.90$, whence $\sigma = 1.7$ This measure of dispersion gives more weight to extreme cases than the other measures of variability. Although it is more tedious to calculate, it is more generally used.

NOTE. Whenever the curve is symmetric about a central value which is simultaneously mean, mode, and median, then Q, the quartile deviation, becomes identical with the median deviation or probable

error. It is usually about 5/6 of the $A.D.$ or arithmetic mean of the deviations. The standard deviation σ includes, in an ordinary case, on each side of the mean *approximately* 1/3 of all the measurements. Hence approximately 2/3 of all the measurements are included between the two values. In the above example approximately 2/3 of all the measurements lie within the range 11 ± 1.7

326. Coefficient of Variation. The deviation from the mean or median is the ***index of variability*** and expresses the tendency of the data to cluster about the mean. The quotient d/A is called a coefficient of ***relative variability.*** The quantity $C = 100$ times the standard deviation divided by the mean which gives the ratio of the standard deviation to the mean expressed in per cent is called the ***coefficient of variation.***

EXERCISES

1. In an examination twenty questions were given and each answer marked right or wrong. A tabulation of the grades showed the following.

Grade	60	65	70	75	80	85	90	95	100
Frequency	3	8	15	32	54	75	70	36	6

Find the median, quartiles, and quartile deviation.

2. In a mental test given to 147 college Freshmen the following scores were made.

Score	120	130	140	150	160	170	180	190	200
Frequency	1	3	6	15	32	50	28	10	2

Find (*a*) the mean deviation; (*b*) the standard deviation; (*c*) the coefficient of variation.

3. In a test the examiner read a list of words. After the reading of each word time was given for each person to write down other words associated in some way with the word read. The following table shows the record made by a group of men and a group of women.

No. of Associations.....	2	4	6	8	10	12	14	16	18	20
Frequencies (Men).....	2	8	28	52	58	36	32	20	6	2
Frequencies (Women)....	0	4	16	40	64	72	52	30	12	4

Compare their coefficients of variation.

4. An angle was measured one hundred times with an average of 60″. The errors or deviations found by subtracting each measurement from the average value were distributed as follows.

Between	+6″ and +5″	+5″ and +4″	+4″ and +3″	+3″ and +2″	+2″ and +1″	+1″ and 0″	0″ and −1″	−1″ and −2″	−2″ and −3″	−3″ and −4″	−4″ and −5″
No. of Errors..	1	2	4	7	12	24	25	13	6	4	2

Find the probable error.

327. The Binomial Distribution. In the expanded form of $(a + b)^n$, the coefficients are symmetrically arranged as shown by Pascal's triangle. If a coin is tossed three times it may fall so as to give any one of the combinations, three heads, two heads and one tail, one head and two tails, three tails. The probability of any one of these different ways is given by a term in the expansion, by the binomial theorem, of $(1/2 + 1/2)^3$. Expanding, we find

$$\left(\frac{1}{2}+\frac{1}{2}\right)^3 = \frac{1}{2^3}(1 + 3 + 3 + 1)$$

$$= \frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8}.$$

The fractions show the probability of 3 heads, 2 heads, 1 head and 0 heads, respectively.

If 10 coins are tossed, there are 11 possible results: 10 heads, 9 heads and 1 tail, 8 heads and 2 tails, $\cdots$ 1 head and 9 tails, 10 tails. Since each of these 10 coins can fall in

either of two ways, ten coins can fall in 2^{10} ways. Of the 1024 ways, only one way gives all heads, 10 ways give 9 heads and one tail, 45 ways give 8 heads and 2 tails, etc.

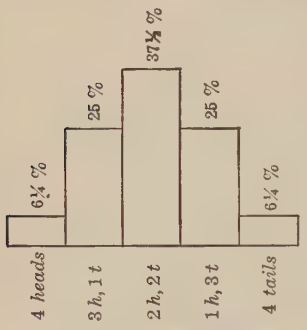


Fig. 180.

The number of ways any particular combination can occur, say 6 heads and 4 tails, is equal to the coefficient of h^6t^4 in the expansion

$$(h + t)^{10} = h^{10} + 10h^9t^1 + 45h^8t^2 + 120h^7t^3 + 210h^6t^4 + \cdots.$$

Since the sum of all the coefficients is 1024 and 10 heads can occur in only one way, its probability is $1/1024 = 0.1\%$ approximately. Nine heads and one tail can occur in 10 ways; hence its probability is $10/1024 = 1\%$ approximately. The probability of eight heads and two tails is $45/1024 = 4.4\%$ approximately. If n coins are tossed, Fig. 180 shows, for $n = 4$, the probability of r heads and s tails, where $r + s = n$.

The distribution of frequencies shown in the figure above is called a *binomial distribution*. It should be noted that in each binomial distribution the sum of all the probabilities is one; hence the sum of the heights of all the rectangles is unity. As the number of terms in the binomial distribution is increased the number of rectangles is increased. The staircase, as n is indefinitely increased, and

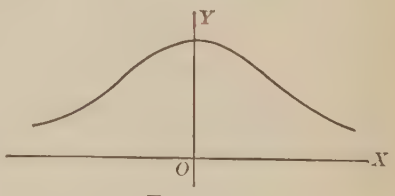


Fig. 181.

the common length of base of the intervals is decreased in exactly the right manner, approaches the curve

$$y = ke^{-h^2x^2}. \tag{9}$$

The proof of this statement will not be given here.

This curve is variously known as the ***error curve,*** the ***probability curve,*** the ***normal distribution curve,*** or the ***normal frequency curve.*** The equation $y = ke^{-h^2x^2}$ is the most important equation in the theory of errors, and should be committed to memory. In graphic representation of data the width of the base of the rectangle represents the least count of the instrument, that is, the unit of measurement, while the altitude represents the frequency of that measurement. In comparing different sets of measurements the curve will be found to vary from the low flat form to the high narrow form. If the deviation (Q; $A.D$; or σ) is large, the curve resembles B. If the deviation is small (that is, the measurements lie very near the mean), the curve takes the form marked A. The form A indicates greater precision.

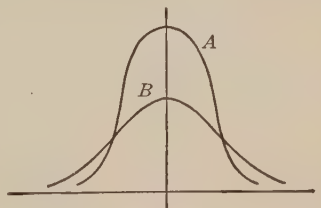


FIG. 182.

The symmetric distribution as represented in the probability curve is commonly found in probability, physical, chemical, and biological measurements. The moderately asymmetric distribution in which the frequencies decrease more rapidly on one side of the value of the maximum frequency than on the other is met with sufficiently often, however, to require consideration in the theory of statistics.

Since the probabilities of any two errors x_1 and x_2 are to each other as the heights of the corresponding rectangles, they are to each other as y_1 is to y_2, or as $e^{-h^2x_1^2}$ is to $e^{-h^2x_2^2}$.

328. Method of Least Squares. If an object is measured several times, the measures being $M_1, M_2, \cdots, M_n$, and Z is the most probable value, then the errors, which we will represent by $x_1, x_2, \cdots, x_n$, are $Z-M_1, Z-M_2, \cdots, Z-M_n$. Now the probability of the first error x_1 is $ke^{-h^2x_1^2}$, of the second error $ke^{-h^2x_2^2}$, etc. Hence the probability of the simultaneous occurrence of this set of errors is the product of the probabilities that each will be made separately, that is,

$$(10) \qquad \begin{aligned} P &= k^n e^{-h^2x_1^2} e^{-h^2x_2^2} \cdots e^{-h^2x_n^2} \\ &= k^n e^{-h^2(x_1^2+x_2^2+\cdots+x_n^2)}. \end{aligned}$$

The most probable set of errors or residuals is that for which P is largest, hence for which $x_1^2 + x_2^2 + \cdots + x_n^2$ is least; that is, $(Z - M_1)^2 + (Z - M_2)^2 + \cdots + (Z - M_n)^2$ is least. From the principle of maxima and minima, the derivative is zero when the quantity is least. Hence

$$(11) \qquad 2(Z - M_1) + 2(Z - M_2) + \cdots + 2(Z - M_n) = 0,$$

or

$$Z = \frac{M_1 + M_2 + \cdots + M_n}{n}.$$

That is, the most probable value of n equally precise measurements is the arithmetic mean of the measurements. Thus, the most probable value of the area of a circle as determined from measurements of the diameter is $\pi/4$ times the square of the arithmetical mean of the results of those measurements.

PRINCIPLE. *The most probable value of a measured quantity that can be deduced from a series of direct observations, made with equal care and skill, is that for which the sum of the squares of the residuals is a minimum.*

329. The Probability Curve. Symmetry. Since the equation $y = ke^{-h^2x^2}$ remains unchanged when x is replaced by $-x$, the curve is symmetric with respect to the y-axis.

INTERCEPT. The y intercept or maximum ordinate is the quantity k, since $y = k$ when $x = 0$.

ASYMPTOTE. The x-axis is an asymptote. As x increases indefinitely in absolute value, y approaches zero as a limit.

330. The Value of k. If we imagine all the vertical strips placed end to end, the total length is 1 (that is, $\Sigma y = 1$), since this is the sum of the probabilities of all possible errors. See Figs. 180, 181, and 182. It is possible to pass to the limit in such a way that the area under the limiting curve is also unity. If this area be found by integration, it can be shown that

$$A = k\int_{-\infty}^{\infty} e^{-h^2x^2}dx = k\frac{\sqrt{\pi}}{h} = 1. \tag{12}$$

The integral in (12) is beyond the scope of this book, but we shall assume that the result stated is correct. Then we have

$$k = \frac{h}{\sqrt{\pi}}. \tag{13}$$

Substituting, we find

$$y = \frac{h}{\sqrt{\pi}}e^{-h^2x^2}. \tag{13a}$$

If two sets of measurements are compared, it will be found that the measurements in the more precise set will cluster more closely about the mean. The value of h in this set, since it is proportional to the y-intercept, will be larger. For this reason the value of h is often called the ***measure of precision*** or the ***precision index*** of a set of measurements.

331. Approximate Formula for the Precision Index. Let there be n observations made upon an unknown quantity, the errors being $x_1, x_2, \cdots, x_n$; then the probabilities of these errors are, respectively,

$$y_1 = \frac{h}{\sqrt{\pi}}e^{-h^2x_1^2},\ y_2 = \frac{h}{\sqrt{\pi}}e^{-h^2x_2^2},\ \cdots,\ y_n = \frac{h}{\sqrt{\pi}}e^{-h^2x_n^2}. \tag{14}$$

The probability of the given set of errors is the product of the probabilities of the separate errors y, whence

$$y = \left(\frac{1}{\sqrt{\pi}}\right)^n h^n e^{-h^2\Sigma x^2}. \tag{15}$$

If the product y is a maximum, it can be shown that we shall have

$$\frac{dy}{dh} = \left(\frac{1}{\sqrt{\pi}}\right)^n e^{-h^2\Sigma x^2}\left[nh^{n-1} - 2h^{n+1}\Sigma x^2\right] = 0, \tag{16}$$

which we shall here assume to be correct. From (16) it follows that

$$h^2 = \frac{n}{2\Sigma x^2}. \tag{17}$$

This value of h would be adequate if the true errors x were known, and does very well in any case. However, if we change from errors to deviations, the denominator is diminished, since $\Sigma d^2 < \Sigma x^2$ (the sum of the squares of the deviations from the mean is a minimum). In practice it is assumed that the best value of the numerator to conform to this change is $n - 1$. The formula for h now becomes

$$h = \sqrt{\frac{n-1}{2\Sigma d^2}}, \quad \text{or} \quad \frac{1}{h} = \sqrt{\frac{2\Sigma d^2}{n-1}}. \tag{18}$$

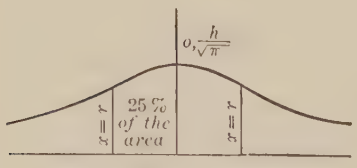


Fig. 183.

332. Probable Error. The probable error is a measure of the deviation of the measurements from the true value. The value r of x such that half the errors lie between the limits $+ r$ and $- r$ from the axis of y is called the probable error. Graphically it is the abscissa of the ordinate

which bisects the area under one branch of the curve. If the distribution is symmetrical, the mean and median coincide and the probable error is the same as the quartile deviation, namely, $(Q_3 - Q_1)/2$.

The area under the curve $y = (h/\sqrt{\pi})e^{-h^2x^2}$ from $-hx$ to $+hx$ is given in the following table for different values of hx.

hx	A
0.0	0.0000
0.1	0.1125
0.2	0.2227
0.3	0.3286
0.4	0.4284
0.5	0.5205
0.6	0.6039

hx	A
0.7	0.6778
0.8	0.7421
0.9	0.7969
1.0	0.8427
1.1	0.8802
1.2	0.9103
1.3	0.9340

hx	A
1.4	0.9523
1.5	0.9661
1.6	0.9763
1.7	0.9838
1.8	0.9891
1.9	0.9928
2.0	0.9953

hx	A
2.1	0.9970
2.2	0.9981
2.3	0.9989
2.4	0.9993
2.5	0.9996
2.6	0.9998

From the table it is seen that A is 1/2 for some value of hx between 0.4 and 0.5 By carrying the computation to more decimal places it has been found that $hr = 0.4769$, whence $r = (0.4769/h)$. Substituting the value of h from § 331,

$$r = 0.4769\sqrt{\frac{2\Sigma d^2}{n-1}}$$

$$= 0.6745\sqrt{\frac{\Sigma d^2}{n-1}} \quad \text{or} \quad 0.6745\sqrt{\frac{\Sigma x^2}{n}}$$

$$= 0.6745\sigma, \text{ approximately.}$$

Hence one-half of the total frequencies of any normal distribution lie between $x = 0.6745\sigma$ and $x = -0.6745\sigma$. This value r of x is called the probable error of a single observation.

NOTE. The probable error is one of the most valuable tools of the statistician or the scientific investigator.

EXERCISES

1. If a coin is tossed 6 times, find by means of the binomial formula the chance of (*a*) all heads, (*b*) five heads and one tail, (*c*) four heads and two tails.

2. Draw the frequency distribution polygon for the expansion $(\frac{1}{2} + \frac{1}{2})^5$.

3. The following grades were obtained in a spelling test.

Grade	60	65	70	75	80	85	90	95	100
Frequency	1	2	7	20	25	30	22	10	3

Construct the frequency polygon.

4. In Ex. 1 there are seven possibilities (i.e., all heads, etc.). Show that the sum of the seven probabilities is one.

5. If one hundred coins be thrown up, find the chance of all heads.

6. If two dice are thrown, find the probability of a double six. Of getting only one six. Of not getting a six.

7. If three dice are thrown, find the probability of (*a*) three sixes, (*b*) two sixes and some other number, (*c*) not getting a six.

8. Draw the graph of the probability curve $y = e^{-x^2}$.

9. A surveyor made the following readings. Find the mean and the probable error.

50.41, 50.42, 50.405, 50.421, 50.408, 50.407

10. A line is measured five times; the measurements are

20.31, 20.32, 20.309, 20.308, 20.314

Find the mean and the probable error.

CHAPTER XIX

COMPLEX NUMBERS

333. The Real Number System. Elementary arithmetic deals with positive rational numbers only, that is, such numbers as 0, 1/2, 3/4, 1, 2, 3, $\cdots$. Hence it is impossible to subtract 7 from 4, that is, the larger from the smaller. In algebra the unknown x is allowed to assume negative values as well as positive, which makes possible the representation of reversed quantities when dealing with coordinates, assets and debts, gains and losses, latitudes north or south, elevations above or below a datum plane, temperature above or below zero. This doubles the field of arithmetical numbers and makes subtraction possible in all cases. Hence, we may write $4 - 7 = -3$.

Historically, positive integers were the first numbers used, fractions appeared next, and the negative numbers were a much later development. Numbers of these three types are known as rational numbers. The real number system includes the above named types together with their limits, namely, such irrational numbers as $\sqrt{3}$, $\sqrt[3]{10}$, $-2 \pm \sqrt{5}$; and such transcendental numbers as $\pi = 3.1416 \cdots$, and $e = 2.718 \cdots$. One property of real numbers is that the square of any real number except zero is a positive number.

The complete set of positive and negative numbers, called the system of real numbers, can be represented by all the points on a line extending in both directions from the zero point.

334. Complex Numbers. In the solution of quadratic equations, not all roots can be expressed as real numbers. If $x^2 - 4x + 10 = 0$, x is neither a positive nor a negative

number, and cannot belong to the system of real numbers. It is desirable, therefore, to extend the system and indeed in such a way as to include numbers of the form $a \pm bi$, where a and b are real numbers and i is a symbol for a number whose square is -1. If $x^2 = -1$, $x = \pm\sqrt{-1}$, written $\pm i$. Likewise if $x^2 = -9$, $x = \pm\sqrt{-9} = \pm 3\sqrt{-1}$, written $\pm 3i$. The symbol $i = \sqrt{-1}$ is called the imaginary unit, and a number of the form $\sqrt{-b}$, where $-b$ is a negative number or zero, is called a pure imaginary number. The sum of a real and pure imaginary number, as $a \pm bi$, is called a complex number. Hence complex numbers include as special cases all real numbers and all pure imaginary numbers. Any non-real complex number is often called imaginary.

With the introduction of complex numbers, what is called the Number System of Algebra is complete and is sufficient for solving all algebraic equations not only when the coefficients are real, but also when these in turn are imaginary.

Numbers like $\pm b\sqrt{-1}$ and $a \pm bi$ do not belong to the system of real numbers just as a fraction, a negative number, or an irrational number does not belong to a number system consisting of positive integers only. Two imaginary numbers which differ only in the sign of the term containing $\sqrt{-1}$, as $x + iy$ and $x - iy$, are said to be conjugate imaginary.

335. Graphic Solution. Let us consider the quadratic equation $(x - a)^2 + k = 0$, where k may have the values $-d$, 0, or d, d being a positive real number. It is easily shown that the roots of the equation:

(1) $(x - a)^2 - d = 0$, are $(a \pm \sqrt{d})$.

(2) $(x - a)^2 = 0$, are $(a \pm 0)$.

(3) $(x - a)^2 + d = 0$, are $a \pm i\sqrt{d}$.

If the graph of $y = (x - a)^2 + k$ is plotted, the resulting

curve is a parabola. In case (1) the vertex of the parabola is at $(a, -d)$; in (2) at (a, o); in (3) at (a, d). (See Fig. 184.)

The roots of (3) are complex but may be read from the graph. Let ON be the abscissa of the lowest point on the curve and NM the corresponding ordinate. Take $KN = 2MN$ and draw KL parallel to the x-axis. Then the solution is $x = ON + KLi$, or $a \pm i\sqrt{d}$.

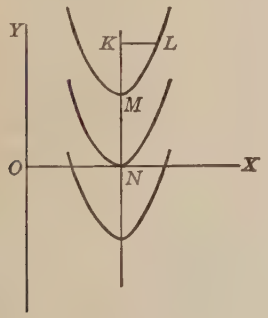


Fig. 184.

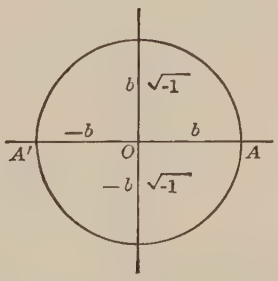


Fig. 185.

336. Representation of Complex Numbers. Suppose a straight line b units long, is rotated about O from a position OA through 180°. It then has the negative direction OA' and graphically represents $-b$. After a further turn of 180° it again takes the position OA and is graphically represented by $+b$. In each instance the rotation through 180° has been equivalent to the multiplication by -1. Since -1 is the product of $\sqrt{-1} \times \sqrt{-1}$, the multiplication by the factor $\sqrt{-1}$ may be taken as equivalent to rotating the line through 90°.

Hence a definite geometrical meaning may be assigned to such expressions as $\pm b\sqrt{-1}$, or $a \pm b\sqrt{-1}$.

Draw a pair of perpendicular axes $X'X$ and $Y'Y$ intersecting at O. The line $X'X$ may be taken as the axis of "reals" and the line $Y'Y$ as the axis of imaginaries.

To represent the complex number $x + iy$, locate the point (x, y) in the plane in the usual way. Since this point represents a distance of x units along the axis of "reals" and a distance of y along the axis of imaginaries, it may be written $x + iy$. If $y = 0$, the number is real and is represented graphically by a point on the x-axis. If $x = 0$, the number is a pure imaginary, and is represented graphically by a point on the y-axis.

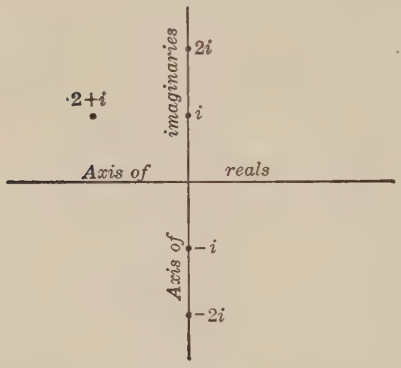


Fig. 186.

Historical Note. The method of representing complex numbers by points in a plane was presented to the Danish Academy in 1797 by Caspar Wessel, a Norwegian. In 1806 the method was rediscovered by J. R. Argand, a Frenchman, and is usually known by the name of Argand's Diagram. The method seems to have received no attention until 1831 when it was rediscovered by Gauss. The symbol i to represent the imaginary unit was introduced by Euler.

The polar form of complex numbers was first used by the great French mathematician Cauchy. Recently the practical importance of complex numbers has been generally recognized by physicists and engineers in dealing with problems of electricity, and steam turbines. It is also used in connection with vectors, as well as in many branches of higher mathematics.

337. Polar Form of Complex Numbers. It is often convenient to represent a complex number by its polar form. To do this connect the point P, which represents $x + iy$,

with the origin, Fig. 187, and let the length of the line be r and the angle XOP be θ. Then

(4) $x = r\cos\theta, \quad y = r\sin\theta, \quad$ and $\quad x^2 + y^2 = r^2.$

Hence the number $x + iy$ may be written in the form

(5) $r\cos\theta + i(r\sin\theta),$

which reduces to

(6) $r(\cos\theta + i\sin\theta)$

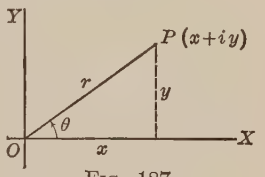


FIG. 187.

This form is called the ***polar form*** of the complex number $x + iy$.

The length $r = \sqrt{x^2 + y^2}$ is called the radius, modulus, or absolute value, and the angle $XOP(\theta)$ measured counterclockwise from the axis of reals to the line OP is called the angle, or argument, of the complex number.

338. Equal Complex Numbers. Two expressions, $a + bi$ and $c + di$, can represent the same complex number if, and only if, their real parts are equal and also their imaginary parts.

If $a + bi = c + di$, then

$$(a - c)^2 = i^2(d - b)^2, \quad \text{or} \quad (a - c)^2 = -(d - b)^2.$$

The quantity on the left being not less than zero and that on the right being not greater than zero, the two cannot be equal unless each is equal to zero. Hence $a = c$, and $b = d$.

Geometrically, a given complex number can be represented by one and only one point.

EXERCISES

1. Represent graphically each of the numbers.

(*a*) 2.	(*c*) $3i$.	(*e*) $4i$.	(*g*) -6.	(*i*) i.
(*b*) -3.	(*d*) $-5i$.	(*f*) $-7i$.	(*h*) $-6i$.	(*j*) $-8i$.

2. Represent graphically each of the following powers of i:

$$i,\ i^2,\ i^3,\ i^4,\ i^6,\ i^8,\ i^9,\ i^{16},\ i^{25},\ i^{30}.$$

3. Locate each of the following points.

(*a*) $4 + 2i$.	(*e*) $6 + 2i$.	(*i*) $5 + 3i$.	(*m*) $5 - 6i$.
(*b*) $6 + i$.	(*f*) $8 + 4i$.	(*j*) $1 + 6i$.	(*n*) $-8 - 2i$.
(*c*) $-5 + 2i$.	(*g*) $-6 + 3i$.	(*k*) $-4 + 5i$.	(*o*) $6 - 5i$.
(*d*) $-7 + 4i$.	(*h*) $-5 + 5i$.	(*l*) $-6 + 7i$.	(*p*) $-3 - 8i$.

4. Represent graphically each of the following numbers and find the absolute value and the angle. Write the numbers in the polar form. (*a*) $4 + 3i$. The point is represented by P in Fig. 188. The absolute value $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$. The polar form is $5(\cos\theta + i\sin\theta)$, where $\cos\theta = 4/5$ and $\sin\theta = 3/5$.

(*b*) $6 + 8i$.	(*e*) $4 + 3i$.	(*h*) $1 + i$.	(*k*) i.
(*c*) $4 - 3i$.	(*f*) $8 - 6i$.	(*i*) $5 + 2i$.	(*l*) 6.
(*d*) $12 + 5i$.	(*g*) $2 + 3i$.	(*j*) $1 - i$.	(*m*) -8.

5. Write each of the following complex numbers in the form $x + iy$.

(*a*) $5(\cos 30° + i \sin 30°)$.	(*d*) $10(\cos 180° + i \sin 180°)$.
(*b*) $2(\cos 45° + i \sin 45°)$.	(*e*) $8(\cos 90° + i \sin 90°)$.
(*c*) $4(\cos 60° + i \sin 60°)$.	(*f*) $3(\cos 0° + i \sin 0°)$.

6. What must be the values of x and y in order that the following equations may be true.

(*a*) $x + y + i(x - 2y) = 5 - 4i$.	(*c*) $x^2 + y^2 + i(x - y) = 17 - 3i$.
(*b*) $2x + 3y + i(x - y) = 9 + 2i$.	(*d*) $x^2 + y^2 + i(2x - y) = 5 + 4i$.

7. Show that all even powers of i are either ± 1, and all odd powers of i are either $\pm i$.

8. Solve the following quadratic equations graphically and check the solution by means of the formula.

(*a*) $(x - 2)^2 + 4 = 0$.	(*d*) $x^2 - 2x + 5 = 0$.	(*g*) $x^2 + 2x + 17 = 0$.
(*b*) $(x - 4)^2 + 9 = 0$.	(*e*) $x^2 - 4x + 13 = 0$.	(*h*) $x^2 + 9 = 0$.
(*c*) $(x - 6)^2 + 4 = 0$.	(*f*) $x^2 - 2x + 2 = 0$.	(*i*) $x^2 + 4 = 0$.

339. Addition and Subtraction of Complex Numbers. We assume that the laws of algebraic combination, if suitably stated, can be extended from their application to real numbers to apply without modification to complex numbers. For example, in adding two numbers, we may introduce

parentheses in the manner already familiar, etc. Hence

$$a + bi + c + di = a + c + (b + d)i,$$

and

$$a + bi - (c + di) = (a - c) + (b - d)i.$$

Hence to add (subtract) complex numbers, add (subtract) the real and imaginary parts separately.

To add two complex numbers $a + bi$ and $c + di$ graphically, connect the points P_1 and P_2, which represent these numbers, with the origin O and complete the parallelogram. The new vertex P represents the sum of the two given complex numbers. For

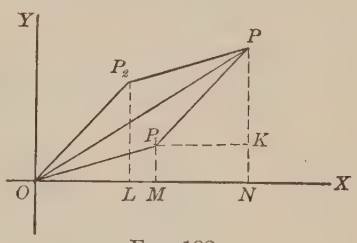


Fig. 188.

$$ON = OM + MN = OM + P_1K = OM + OL = a + c.$$

Also

$$NP = NK + KP = MP_1 + LP_2 = b + d.$$

Since P has the coordinates $(a + c, b + d)$, it represents the complex number $(a + c) + (b + d)i$ which is the sum of $a + bi$ and $c + di$.

To subtract one complex number $c + di$ from another $a + bi$ graphically, we plot the points $a + bi$ and $-c - di$ and proceed as in addition.

EXERCISES

Perform the following operations algebraically and graphically.

1. $3 + i + (1 + 4i)$.
2. $5 + 2i + (2 + 4i)$.
3. $6 + 3i + (1 + 5i)$.
4. $3 + 4i + (2 + 2i)$.
5. $5 + 3i + (-4 + 2i)$.
6. $3 + 2i - (1 - 4i)$.
7. $6 + i + (3i)$.
8. $4 + 5i + (6)$.
9. $5 + i - (3 + 5i)$.

10. Prove geometrically that the sum of two conjugate complex numbers is real and that their difference is a pure imaginary number.

11. A horizontal force of 24 lb. and a vertical force of 10 lb. act simultaneously on a point. Represent the resulting force by a complex number.

12. A ship is sailing due east at the rate of 12 miles per hour. A sailor crosses the deck from south to north at the rate of 5 miles per hour. Represent the resulting velocity of the sailor as a complex number.

340. Multiplication of Complex Numbers. Let $a + bi$ and $c + di$ be any two complex numbers. Since i obeys the laws of algebra, $(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$ which is a complex number.

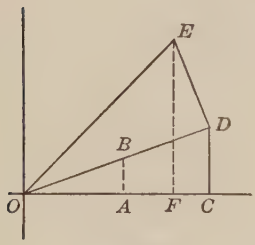


FIG. 189.

To multiply two complex numbers $3 + i$ and $2 + i$ graphically, we may obtain a number from $3 + i$ in the same way that $2 + i$ is obtained from the positive real unit.

Hence take $3 + i$ (that is, OB) twice; then take the length $3 + i$ at right angles to OD (that is, DE) once. The line joining the point E to the origin is the algebraic product of the two factors, namely, $5 + 5i$.

This work is often simplified by reducing each factor to the polar form. Let

$$a + bi = r_1(\cos \theta_1 + i \sin \theta_1), \tag{7}$$

and

$$c + di = r_2(\cos \theta_2 + i \sin \theta_2). \tag{8}$$

By actual multiplication, we find

$$(a + bi)(c + di) = r_1 r_2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2). \tag{9}$$

Since

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1+\theta_2),$$

and

$$\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2),$$

we have

$$(10)\qquad r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$

But this is the polar form of the number which has the absolute value r_1r_2 and the angle $\theta_1 + \theta_2$.

Hence the product of two complex numbers is another complex number whose absolute value is the product of the absolute values and whose angle is the sum of the angles of the numbers.

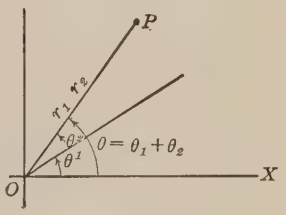


FIG. 190.

Hence, to construct a line which represents the product draw through O a line making an angle $\theta = \theta_1 + \theta_2$ with the line OX, and construct on this line a segment OP whose length is r_1r_2.

341. De Moivre's Theorem. Expressions of the form $r(\cos\theta + i\sin\theta)$ play an important part in modern analysis. If in (10) we put $r_1 = r_2 = r$, and $\theta_1 = \theta_2 = \theta$, we have

$$(11)\qquad r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta).$$

Multiplying both sides of this identity by

$$r(\cos\theta + i\sin\theta),$$

we obtain

$$(12)\qquad r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta).$$

Similarly,

$$(13) \qquad r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta),$$

where n is any positive integer. Hence, the nth power of the complex number $r(\cos\theta + i\sin\theta)$ has r^n for its absolute value and $n\theta$ for its angle. This theorem, known as De Moivre's theorem, is also true when n is a negative integer or a rational fraction.

342. De Moivre's Theorem Extended. If n is a positive integer as in § 341,

$$(14) \qquad \cos\theta + i\sin\theta = (\cos\theta/n + i\sin\theta/n)^n.$$

Therefore

$$(15) \qquad (\cos\theta + i\sin\theta)^{1/n} = \cos\theta/n + i\sin\theta/n.$$

Since θ may be increased by any integral multiple of 2π without changing the value of $\cos\theta + i\sin\theta$, the following n expressions are the nth roots of $r(\cos\theta + i\sin\theta)$.

$$r^{1/n}(\cos\theta/n + i\sin\theta/n), \quad r^{1/n}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right),$$

$$r^{1/n}\left(\cos\frac{\theta+4\pi}{n} + i\sin\frac{\theta+4\pi}{n}\right),$$

$$\cdots r^{1/n}\left(\cos\frac{\theta+(n-1)2\pi}{n} + i\sin\frac{\theta+(n-1)2\pi}{n}\right).$$

There are only n such values, since an equation of the nth degree cannot have more than n roots.

Hence, if n is any positive integer,

$$(16) \qquad [r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\frac{(\theta+k2\pi)}{n} + i\sin\frac{(\theta+k2\pi)}{n}\right],$$

where k is a positive integer or 0.

Now if k takes the successive values $0, 1, 2, \cdots, (n-1)$, we find n different results, each one of which raised to the nth power equals $r(\cos\theta + i\sin\theta)$. From this we may state the following result.

THEOREM. *Any number, except zero, has n distinct nth roots.*

HISTORICAL NOTE. Abraham De Moivre (1667–1754) was a member of a French Protestant family. At the revocation of the Edict of Nantes, when he was eighteen, he took refuge in England, where he became the close personal friend of Sir Isaac Newton. He discovered the theorem which bears his name.

343. Roots of Unity. Let us consider the cube roots of unity. The modulus of 1 is 1, and the argument is 0°.

Therefore,

$$1 = 1[\cos(0° + k360°) + i\sin(0° + k360°)].$$

Hence the cube roots of unity are

$$1\left[\cos\frac{0° + k360°}{3} + i\sin\frac{(0° + k360°)}{3}\right].$$

Letting k take the successive values 0, 1, 2, we obtain, for the three cube roots of 1, the following values:

$1(\cos 0° + i\sin 0°)$, that is, 1.

$1(\cos 120° + i\sin 120°)$, that is, $(-1 + i\sqrt{3})/2$.

$1(\cos 240° + i\sin 240°)$, that is, $(-1 - i\sqrt{3})/2$.

EXERCISES

Multiply both algebraically and graphically, and find the absolute value and the angle of each of the following products.

1. $2 + 2i$ by $3 + 3i$.
2. $4 + i$ by $1 + 2i$.
3. $4 + 2i$ by 3.
4. $3 + 4i$ by i.
5. $6 + 2i$ by $2 + 2i$.
6. $3 + i$ by $1 + i\sqrt{3}$.
7. Show that $(1 + i)^8 = 16$.

SUGGESTION. $1 + i = \sqrt{2}(\cos 45° + i \sin 45°)$.

8. Show that $(\sqrt{3} + i)^{12} = 2^{12}$.

SUGGESTION. $\sqrt{3} + i = 2(\cos 30° + i \sin 30°)$.

9. Find the four fourth roots of unity.

10. Find the cube roots of 8.

11. Show that the sum of the roots of the equation $x^3 - 1 = 0$ is zero.

12. Find the six roots of $x^6 = 2^6$.

13. Using De Moivre's theorem, find the indicated roots in each of the following expressions.

(*a*) $[27(\cos 90° + i \sin 90°)]^{1/3}$.
(*b*) $[32(\cos 50° + i \sin 50°)]^{1/5}$.
(*c*) $[\cos 360° + i \sin 360°]^{1/10}$.
(*d*) $[\cos 360° + i \sin 360°]^{1/8}$.

14. One root of $x^3 - 7x^2 + 20x - 50 = 0$ is $1 + 3i$. Find the other roots.

Angle	Sin	Cos	Tan	Ctn	Sec	Csc	
0°	.0000	1.0000	.0000	—	1.0000	—	90°
1	.0175	.9998	.0175	57.2900	1.0002	57.2987	89
2	.0349	.9994	.0349	28.6363	1.0006	28.6537	88
3	.0523	.9986	.0524	19.0811	1.0014	19.1073	87
4	.0698	.9976	.0699	14.3007	1.0024	14.3356	86
5°	.0872	.9962	.0875	11.4301	1.0038	11.4737	85°
6	.1045	.9945	.1051	9.5144	1.0055	9.5668	84
7	.1219	.9925	.1228	8.1443	1.0075	8.2055	83
8	.1392	.9903	.1405	7.1154	1.0098	7.1853	82
9	.1564	.9877	.1584	6.3138	1.0125	6.3925	81
10°	.1736	.9848	.1763	5.6713	1.0154	5.7588	80°
11	.1908	.9816	.1944	5.1446	1.0187	5.2408	79
12	.2079	.9781	.2126	4.7046	1.0223	4.8097	78
13	.2250	.9744	.2309	4.3315	1.0263	4.4454	77
14	.2419	.9703	.2493	4.0108	1.0306	4.1336	76
15°	.2588	.9659	.2679	3.7321	1.0353	3.8637	75°
16	.2756	.9613	.2867	3.4874	1.0403	3.6280	74
17	.2924	.9563	.3057	3.2709	1.0457	3.4203	73
18	.3090	.9511	.3249	3.0777	1.0515	3.2361	72
19	.3256	.9455	.3443	2.9042	1.0576	3.0716	71
20°	.3420	.9397	.3640	2.7475	1.0642	2.9238	70°
21	.3584	.9336	.3839	2.6051	1.0711	2.7904	69
22	.3746	.9272	.4040	2.4751	1.0786	2.6695	68
23	.3907	.9205	.4245	2.3559	1.0864	2.5593	67
24	.4067	.9135	.4452	2.2460	1.0946	2.4586	66
25°	.4226	.9063	.4663	2.1445	1.1034	2.3662	65°
26	.4384	.8988	.4877	2.0503	1.1126	2.2812	64
27	.4540	.8910	.5095	1.9626	1.1223	2.2027	63
28	.4695	.8829	.5317	1.8807	1.1326	2.1301	62
29	.4848	.8746	.5543	1.8040	1.1434	2.0627	61
30°	.5000	.8660	.5774	1.7321	1.1547	2.0000	60°
31	.5150	.8572	.6009	1.6643	1.1666	1.9416	59
32	.5299	.8480	.6249	1.6003	1.1792	1.8871	58
33	.5446	.8387	.6494	1.5399	1.1924	1.8361	57
34	.5592	.8290	.6745	1.4826	1.2062	1.7883	56
35°	.5736	.8192	.7002	1.4281	1.2208	1.7434	55°
36	.5878	.8090	.7265	1.3764	1.2361	1.7013	54
37	.6018	.7986	.7536	1.3270	1.2521	1.6616	53
38	.6157	.7880	.7813	1.2799	1.2690	1.6243	52
39	.6293	.7771	.8098	1.2349	1.2868	1.5890	51
40°	.6428	.7660	.8391	1.1918	1.3054	1.5557	50°
41	.6561	.7547	.8693	1.1504	1.3250	1.5243	49
42	.6691	.7431	.9004	1.1106	1.3456	1.4945	48
43	.6820	.7314	.9325	1.0724	1.3673	1.4663	47
44	.6947	.7193	.9657	1.0355	1.3902	1.4396	46
45°	.7071	.7071	1.0000	1.0000	1.4142	1.4142	45°
	Cos	Sin	Ctn	Tan	Csc	Sec	Angle

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

For more extended tables, see *The Macmillan Tables.*

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Table III.—Compound Interest: $(1 + r)^n$

Amount of One Dollar Principal at Compound Interest After n Years

n	2 %	2½%	3 %	3½%	4 %	4½ %	5 %	6 %	7 %
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700
2	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	1.1449
3	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	1.2250
4	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	1.3108
5	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	1.4026
6	1.1262	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	1.5007
7	1.1487	1.1887	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036	1.6058
8	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	1.7182
9	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	1.8385
10	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	1.9672
11	1.2434	1.3121	1.3842	1.4600	1.5395	1.6229	1.7103	1.8983	2.1049
12	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	2.2522
13	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	2.4098
14	1.3195	1.4130	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609	2.5785
15	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	2.7590
16	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	2.9522
17	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	3.1588
18	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	3.3799
19	1.4568	1.5987	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	3.6165
20	1.4859	1.6386	1.8061	1.9898	2.1911	2.4117	2.6533	3.2071	3.8697
21	1.5157	1.6796	1.8603	2.0594	2.2788	2.5202	2.7860	3.3996	4.1406
22	1.5460	1.7216	1.9161	2.1315	2.3699	2.6337	2.9253	3.6035	4.4304
23	1.5769	1.7646	1.9736	2.2061	2.4647	2.7522	3.0715	3.8197	4.7405
24	1.6084	1.8087	2.0328	2.2833	2.5633	2.8760	3.2251	4.0489	5.0724
25	1.6406	1.8539	2.0938	2.3632	2.6658	3.0054	3.3864	4.2919	5.4274
26	1.6734	1.9003	2.1566	2.4460	2.7725	3.1407	3.5557	4.5494	5.8074
27	1.7069	1.9478	2.2213	2.5316	2.8834	3.2820	3.7335	4.8223	6.2139
28	1.7410	1.9965	2.2879	2.6202	2.9987	3.4297	3.9201	5.1117	6.6488
29	1.7758	2.0464	2.3566	2.7119	3.1187	3.5840	4.1161	5.4184	7.1143
30	1.8114	2.0976	2.4273	2.8068	3.2434	3.7453	4.3219	5.7435	7.6123
31	1.8476	2.1500	2.5001	2.9050	3.3731	3.9139	4.5380	6.0881	8.1451
32	1.8845	2.2038	2.5751	3.0067	3.5081	4.0900	4.7649	6.4534	8.7153
33	1.9222	2.2589	2.6523	3.1119	3.6484	4.2740	5.0032	6.8406	9.3253
34	1.9607	2.3153	2.7319	3.2209	3.7943	4.4664	5.2533	7.2510	9.9781
35	1.9999	2.3732	2.8139	3.3336	3.9461	4.6673	5.5160	7.6861	10.6766
36	2.0399	2.4325	2.8983	3.4503	4.1039	4.8774	5.7918	8.1473	11.4239
37	2.0807	2.4933	2.9852	3.5710	4.2681	5.0969	6.0814	8.6361	12.2236
38	2.1223	2.5557	3.0748	3.6960	4.4388	5.3262	6.3855	9.1543	13.0793
39	2.1647	2.6196	3.1670	3.8254	4.6164	5.5659	6.7048	9.7035	13.9948
40	2.2080	2.6851	3.2620	3.9593	4.8010	5.8164	7.0400	10.2857	14.9745
41	2.2522	2.7522	3.3599	4.0978	4.9931	6.0781	7.3920	10.9029	16.0227
42	2.2972	2.8210	3.4607	4.2413	5.1928	6.3516	7.7616	11.5570	17.1443
43	2.3432	2.8915	3.5645	4.3897	5.4005	6.6374	8.1497	12.2505	18.3444
44	2.3901	2.9638	3.6715	4.5433	5.6165	6.9361	8.5572	12.9855	19.6285
45	2.4379	3.0379	3.7816	4.7024	5.8412	7.2482	8.9850	13.7646	21.0025
46	2.4866	3.1139	3.8950	4.8669	6.0748	7.5744	9.4343	14.5905	22.4726
47	2.5363	3.1917	4.0119	5.0373	6.3178	7.9153	9.9060	15.4659	24.0457
48	2.5871	3.2715	4.1323	5.2136	6.5705	8.2715	10.4013	16.3939	25.7289
49	2.6388	3.3533	4.2562	5.3961	6.8333	8.6437	10.9213	17.3775	27.5299
50	2.6916	3.4371	4.3839	5.5849	7.1067	9.0326	11.4674	18.4202	29.4570

Table IV.—Compound Discount: $1/(1+r)^n$

PRESENT VALUE OF ONE DOLLAR DUE AT THE END OF n YEARS

n	2 %	2½%	3 %	3½ %	4 %	4½ %	5 %	6 %	7 %
1	.98039	.97561	.97087	.96618	.96154	.95694	.95238	.94340	.93458
2	.96117	.95181	.94260	.93351	.92456	.91573	.90703	.89000	.87344
3	.94232	.92860	.91514	.90194	.88900	.87630	.86384	.83962	.81630
4	.92385	.90595	.88849	.87144	.85480	.83856	.82270	.79209	.76290
5	.90573	.88385	.86261	.84197	.82193	.80245	.78353	.74726	.71299
6	.88797	.86230	.83748	.81350	.79031	.76790	.74622	.70496	.66634
7	.87056	.84127	.81309	.78599	.75992	.73483	.71068	.66506	.62275
8	.85349	.82075	.78941	.75941	.73069	.70319	.67684	.62741	.58201
9	.83676	.80073	.76642	.73373	.70259	.67290	.64461	.59190	.54393
10	.82035	.78120	.74409	.70892	.67556	.64393	.61391	.55839	.50835
11	.80426	.76214	.72242	.68495	.64958	.61620	.58468	.52679	.47509
12	.78849	.74356	.70138	.66178	.62460	.58966	.55684	.49697	.44401
13	.77303	.72542	.68095	.63940	.60057	.56427	.53032	.46884	.41496
14	.75788	.70773	.66112	.61778	.57748	.53997	.50507	.44230	.38782
15	.74301	.69047	.64186	.59689	.55526	.51672	.48102	.41727	.36245
16	.72845	.67362	.62317	.57671	.53391	.49447	.45811	.39365	.33873
17	.71416	.65720	.60502	.55720	.51337	.47318	.43630	.37136	.31657
18	.70016	.64117	.58739	.53836	.49363	.45280	.41552	.35034	.29586
19	.68643	.62553	.57029	.52016	.47464	.43330	.39573	.33051	.27651
20	.67297	.61027	.55368	.50257	.45639	.41464	.37689	.31180	.25842
21	.65978	.59539	.53755	.48557	.43883	.39679	.35894	.29416	.24151
22	.64684	.58086	.52189	.46915	.42196	.37970	.34185	.27751	.22571
23	.63416	.56670	.50669	.45329	.40573	.36335	.32557	.26180	.21095
24	.62172	.55288	.49193	.43796	.39012	.34770	.31007	.24698	.19715
25	.60953	.53939	.47761	.42315	.37512	.33273	.29530	.23300	.18425
26	.59758	.52623	.46369	.40884	.36069	.31840	.28124	.21981	.17220
27	.58586	.51340	.45019	.39501	.34682	.30469	.26785	.20737	.16093
28	.57437	.50088	.43708	.38165	.33348	.29157	.25509	.19563	.15040
29	.56311	.48866	.42435	.36875	.32065	.27902	.24295	.18456	.14056
30	.55207	.47674	.41199	.35628	.30832	.26700	.23138	.17411	.13137
31	.54125	.46511	.39999	.34423	.29646	.25550	.22036	.16425	.12277
32	.53063	.45377	.38834	.33259	.28506	.24450	.20987	.15196	.11474
33	.52023	.44270	.37703	.32134	.27409	.23397	.19987	.14619	.10723
34	.51003	.43191	.36604	.31048	.26355	.22390	.19035	.13791	.10022
35	.50003	.42137	.35538	.29998	.25342	.21425	.18129	.13011	.09366
36	.49022	.41109	.34503	.28983	.24367	.20503	.17266	.12274	.08754
37	.48061	.40107	.33498	.28003	.23430	.19620	.16444	.11580	.08181
38	.47119	.39128	.32523	.27056	.22529	.18775	.15661	.10924	.07646
39	.46195	.38174	.31575	.26141	.21662	.17967	.14915	.10306	.07146
40	.45289	.37243	.30656	.25257	.20829	.17193	.14205	.09722	.06678
41	.44401	.36335	.29763	.24403	.20028	.16453	.13528	.09172	.06241
42	.43530	.35448	.28896	.23578	.19257	.15744	.12884	.08653	.05833
43	.42677	.34584	.28054	.22781	.18517	.15066	.12270	.08163	.05451
44	.41840	.33740	.27237	.22010	.17805	.14417	.11686	.07701	.05095
45	.41020	.32917	.26444	.21266	.17120	.13796	.11130	.07265	.04761
46	.40215	.32115	.25674	.20547	.16461	.13202	.10600	.06854	.04450
47	.39427	.31331	.24926	.19852	.15828	.12634	.10095	.06466	.04159
48	.38654	.30567	.24200	.19181	.15219	.12090	.09614	.06100	.03887
49	.37896	.29822	.23495	.18532	.14634	.11569	.09156	.05755	.03632
50	.37153	.29094	.22811	.17905	.14071	.11071	.08720	.05429	.03395

Table V*a*.—Values of e^x

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.00	1.11	1.22	1.35	1.49	1.65	1.82	2.01	2.23	2.46
1	2.72	3.00	3.32	3.67	4.06	4.48	4.95	5.47	6.05	6.69
2	7.39	8.17	9.03	9.97	11.0	12.2	13.5	14.9	16.4	18.2
3	20.1	22.2	24.5	27.1	30.0	33.1	36.6	40.4	44.7	49.4
4	54.6	60.3	66.7	73.7	81.5	90.0	99.5	110.	122.	134.
5	148.	164.	181.	200.	221.	245.	270.	299.	330.	365.

Table V*b*.—Values of e^{-x}

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.00	0.90	0.82	0.74	0.67	0.61	0.55	0.50	0.45	0.41
1	0.37	0.33	0.30	0.27	0.25	0.22	0.20	0.18	0.17	0.15
2	0.14	0.12	0.11	0.10	0.09	0.08	0.07	0.07	0.06	0.06
3	0.05	0.05	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.02
4	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
5	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table VI.—American Experience Mortality Table

Based on 100,000 living at age 10

At Age	Number Surviving	Deaths	At Age	Number Surviving	Deaths	At Age	Number Surviving	Deaths	At Age	Number Surviving	Deaths
10	100,000	749	**35**	81,822	732	**60**	57,917	1,546	**85**	5,485	1,292
11	99,251	746	**36**	81,090	737	**61**	56,371	1,628	**86**	4,193	1,114
12	98,505	743	**37**	80,353	742	**62**	54,743	1,713	**87**	3,079	933
13	97,762	740	**38**	79,611	749	**63**	53,030	1,800	**88**	2,146	744
14	97,022	737	**39**	78,862	756	**64**	51,230	1,889	**89**	1,402	555
15	96,285	735	**40**	78,106	765	**65**	49,341	1,980	**90**	847	385
16	95,550	732	**41**	77,341	774	**66**	47,361	2,070	**91**	462	246
17	94,818	729	**42**	76,567	785	**67**	45,291	2,158	**92**	216	137
18	94,089	727	**43**	75,782	797	**68**	43,133	2,243	**93**	79	58
19	93,362	725	**44**	74,985	812	**69**	40,890	2,321	**94**	21	18
20	92,637	723	**45**	74,173	828	**70**	38,569	2,391	**95**	3	3
21	91,914	722	**46**	73,345	848	**71**	36,178	2,448			
22	91,192	721	**47**	72,497	870	**72**	33,730	2,487			
23	90,471	720	**48**	71,627	896	**73**	31,243	2,505			
24	89,751	719	**49**	70,731	927	**74**	28,738	2,501			
25	89,032	718	**50**	69,804	962	**75**	26,237	2,476			
26	88,314	718	**51**	68,842	1,001	**76**	23,761	2,431			
27	87,596	718	**52**	67,841	1,044	**77**	21,330	2,369			
28	86,878	718	**53**	66,797	1,091	**78**	18,961	2,291			
29	86,160	719	**54**	65,706	1,143	**79**	16,670	2,196			
30	85,441	720	**55**	64,563	1,199	**80**	14,474	2,091			
31	84,721	721	**56**	63,364	1,260	**81**	12,383	1,964			
32	84,000	723	**57**	62,104	1,325	**82**	10,419	1,816			
33	83,277	726	**58**	60,779	1,394	**83**	8,603	1,648			
34	82,551	729	**59**	59,385	1,468	**84**	6,955	1,470			

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ANSWERS TO EXERCISES

If an exercise has several parts marked (*a*), (*b*), etc., or contains a series of questions to be answered, in general the answer to part (*a*) or to the first question only is given. These answers are indicated in the same way or given in the same order as the questions in the exercises are given. Four place tables have been used in computing with natural functions and five place tables in computing with logarithms. In a few places where compound interest was involved, eight place tables were used.

Pages 2–3

1. 0, 7.
2. 2, 2.
3. 2, 3.
4. 0, 2.
5. 0, 3.
6. 2, 3.
7. 0, $\sqrt{10}$.
8. 2, $2\frac{1}{2}$.
9. 3, 1.
10. 2, $2\frac{1}{2}$.

Pages 8–9

10. Four solutions.
11. (*a*) $(5, \pm 5)$, $(-5, \pm 5)$.
12. (2, 3).
13. $(3, 4 \pm \sqrt{3})$.
14. $(0, \pm 3)$.

Pages 13–14

2. (*a*) $(x-3)^2 + (y-4)^2 = 49$.
(*b*) $(x-2)^2 + (y-3)^2 = 25$.

3. (*a*) $c = (13, -8)$; $r = 2$.
(*b*) $c = (8, 7)$; $r = 3$.

4. (*a*) $x^2 + (y-5)^2 = 25$, or $x^2 + (y+5)^2 = 25$.
(*b*) $(x-7)^2 + y^2 = 49$, or $(x+7)^2 + y^2 = 49$.

Pages 15–16

1. 2, 5.
2. 1, 3.
3. 6, 0.
4. 0, 4.
5. 6, − 3.
6. 3, 4.
7. 2, − 5.
8. 0, 4.
9. 3, 4.
10. 5, 1.
11. 4, 1.
12. 7, 11.
13. 24, 5.
14. 3, 2.
15. 8, − 5.
16. 15, 14.
17. 4, − 6.

Page 18

1. $x = 6, 2$. $y = 2, 6$.
2. $x = 5, -2$. $y = 2, -5$.
3. $x = 6, 6$. $y = 6, -6$.
4. $x = 4, 4$. $y = 4, -4$.
5. $x = 0, 2$. $y = 0, 4$.
6. $x = 3, -3$. $y = 1, 1$.
7. $x = 0, 6$. $y = 0, 6$.
8. $x = 0, 36$. $y = 0, -216$.
9. $x = 2, -3$. $y = 4, -1$.
10. $x = 3, 4$. $y = 4, 3$.
11. $x = 0, 5$. $y = 0, 5$.
12. $x = 4, 4$. $y = 4, -4$.
13. $x = 4, -4$. $y = 4, 4$.
14. $x = 5, -5$. $y = 5, 5$.
15. $x = 7, 7$. $y = 7, -7$

Pages 20–23

5. 0.8 second.
13. $2\frac{1}{2}$ lb.
14. $E = \frac{1}{5}W$.
15. 20 in.
16. 20 000 ft.
17. 64 ft.
18. 4096.
19. 10 in.
20. (*a*) 0.3, 1.2
21. 20 ft.
22. 6.67
24. 10, $2\frac{1}{2}$.
25. $R = 108/x^2$.
26. $C = 112/R$.
27. $xy = 12$.
28. 36 miles per hr.
29. 2370. + lb.
30. Doubles P.

Pages 31–32

6. 6.46 mi.
7. 103.93 ft.
8. 226.1 ft.
9. 89. + rd.
10. 2.7321 mi.
11. 26 ft. 6.8 in.

Pages 36–39

1. 341.6 ft.
2. 241.1 ft., 188.4 ft.
3. 12.34 mi., 49.48 mi.
4. 64.89 ft.
5. 48.62 ft.
6. 22.86 ft.
7. 172.6 ft.
8. 134.35 ft.
9. 732.7 ft.
10. 2.66 ft.
11. 44.07 ft.
12. $AC = 23.18$ ft.
$BC = 6.21$ ft.
13. 1610.6 mi.
14. 463.3 ft.
15. 2.75 ft. at first 100 ft.
16. 446.84 ft.
17. (*a*) $a = 45.889$ ft.
$b = 65.536$ ft.
18. 7′ approx.
19. 3628.2 ft. per hr.
20. 13.2 ft., nearly.
22. 1043.8 ft.

Pages 42–43

1. 9° 18.′7
2. 9° 11′.4
3. 19° 17′.
4. 74° 37′ +.
5. 14° 2′ +.
6. 13° 50′ +.
7. 3° 35′.
8. 22° 1′ +.
9. 2.72 in.
11. 30° 8′.
12. 58° nearly.
13. 13.77 ft., 37° 26′.
14. 1031.7 ft.
15. 524.7 ft.
16. 604.5 ft.
17. 17 350 ft.

Pages 46–47

1. (*a*) 81.
2. (*a*) 9.
6. (*a*) 6.
7. 92 900 000.
8. (*a*) 4 240 000 000.
9. (*a*) 2.4×10^5 miles.

Pages 50–51

1. (*a*) 3.21032 **2.** (*a*) 23.21 **3.** (*a*) 9.48998 − 10 **4.** (*a*) $x = 26°$.

Pages 55–57

5. (*a*) 167.67
6. 1018.3 cu. ft.
7. 5436.3 cu. ft.
8. $1295.47
9. 8.3 + min.
10. 3.2 + min.
11. 49 266 000 sq. mi.
12. $937 500 000.
13. 1962.5 tons.
14. 9. + yrs.
15. 2290.2 mi. per hr.
16. 30 in.
17. 2 seconds nearly.
18. 1110 mi. per min.
19. 269 ft.

21. 328.9 ft.
22. 2.38
23. 113.
24. 69.78
25. 339.9
26. 441.8
27. 26.54
28. 3.19
29. 25.41
30. 25.55
31. 9.510
32. − 131.7
33. 0.289 +
34. 294.3
35. 0.6947
36. 0.225
37. 0.270
38. 3.76
39. 14.63 acres.

Pages 61–62

5. $2\frac{1}{2}$ radians.
6. 3960. mi.
7. 1 mi. approx.
8. 24 000 ft.
9. 92.1 ft.
10. 0.0436 in.
11. 192 ft. per sec.
12. 477.5 r.p.m.
13. 2834 mi,
14. 4.42 in.
16. 12.73 in.

Pages 67–68

4. 21 600°, 120π radians.
5. 240°.
7. (*a*) cos 20°.
9. (*a*) sin 60°.
10. (*a*) − sin 30°.
11. (*a*) − sin 60°.
12. (*a*) − sin 45°.
13. (*a*) 2nd.
16. (*a*) − sin 55°.
17. (*a*) − .8660
18. $x = 90°, 270°$.

Pages 73–75

3. 26.45
4. 289.
5. 37.
6. 182.
7. (*a*) 281.7
8. (*a*) 210 sq. rds.
9. (*a*) 17.95 acres.
10. $253.41
11. 19.88 acres.
12. Correct.
13. $128.40
14. 2626.7 sq. ft.
15. $97.62

Pages 78–80

1. $B = 32°, c = 11.54$
2. $C = 69°, b = 32.866$
3. 42° 10′.3, 56 926, 72° 49′.7
4. 567.1 ft.
5. 12 638 ft.
6. 1314.2; 1668.1
8. 39.64 mi.
9. 4.963 mi., 8.53 mi.
10. 4756.3 ft.
11. 39.52 mi.
12. 364.2 ft.
13. 82.00 ft.
14. 1427.4 ft.
15. 11.34 mi.
16. 4.051 mi., 3.232 mi., height = 3.192 mi.
17. 3.196 mi., 2.453 mi., height = 1.2995 mi.

Pages 83–84

1. $b = 9.165, A = 70° 53'.8$
2. $c = 18.03, A = 46° 5'.7$
3. $b = 27.83, A = 51° 3' +.$
4. $a = 26.45, B = 40° 54' -.$
5. 71.2 rds.
6. 35° 6′ nearly.
7. 43.16 lb.
8. 107.99 mi.
9. 90°.
10. 29° 56′, 56° 15′.
11. 13.03, 17.83 +.
12. 1079 + ft.
13. 91 660 000 mi.
14. 1000.1 yds.
15. 0.9646
16. 1095.6 ft.

Pages 87–88

1. (*a*) $2\sqrt{13}$.
2. (*a*) $2\sqrt{10}$.
3. (*a*) 10.
4. 21.
5. From (2, 3) to (5, 8).
7. Parallelogram.
8. Rectangle.
9. (*a*) (4, 6).
10. $5\sqrt{2}$, $5\sqrt{2}$, $2\sqrt{5}$.
11. 12, 9.
17. Sides, 13, 13, $7\sqrt{2}$.

Pages 90–91

1. 3/2; 7/5.
2. 8/3; 5/8.
3. $-5/3$; $-5/7$.
4. 4/3.
5. 2.
6. 3.
7. 1/2.
9. 3.
10. 45°.
11. 30°.
12. 60°.
13. $\dfrac{a-b}{b-c}$.
14. 1/3, 2, $-1/2$.
18. 0°, 90°.
19. 0.
20. 5/13; 12/13.

Pages 92–94

2. (*a*) $y-3=5(x-2)$.
3. (*a*) $2x-y=1$.
4. (*a*) $y=2x+5$.
5. $y-5=\sqrt{3}(x-1)$.
6. $y+4=\dfrac{x-2}{\sqrt{3}}$.
7. $x-y+9=0$.
8. 1, -1.
9. $x=3$.
10. $y=4$.
11. $x=\sqrt{3}(y-4)$.
12. $A=0.6M$
14. $P=-.06h/100+15$.
15. \$147.
16. Yes.
18. 1090 ft. per sec.
19. $P=x/2+3$.
20. $E=2T$.
21. $P=.4343h$.

Pages 96–97

1. (*a*) Slope-intercept.
2. (*a*) $6x-5y=-8$.
3. (*a*) $5x+7y=35$.
4. (*a*) $5x-4y+2=0$.
5. (*a*) $-4/7$.
6. $l=.2w+2$.
7. $33\frac{1}{16}$.
8. Yes.
9. Yes, yes, yes, yes.

Pages 99–100

4. 63° 26′.
5. (*a*) 45°.
6. (*a*) $3x-y=1$.
7. 26° 34′.
8. $2x-y=5$.
9. $x-5y+50=0$.
11. 5/4, $-4/5$.
12. 2/3, $-3/2$.
13. (*a*) 2.
14. (*a*) $5x-y=10$.
15. Yes.
16. Yes.
19. At $x=-1/7$.

Pages 102–103

1. (*a*) $2x+3y=26$.
2. $2x+5y=18$.
3. $2x-y=-9$.
4. $(x-3)^2+(y-4)^2=36$.
7. $\sqrt{53}$.
9. $y^2=8x-16$.
10. $y^2=12x-36$.
11. $y^2=16x$.
12. (2, 7/4).

Pages 105–106

3. $a = 1$.
4. 2, 3.
5. 3, 4.
6. 2, 7.
7. 1, $\frac{2}{3}$.
8. 1, $3\frac{1}{3}$.
9. -5, -6.
10. -1, -6.
11. -1, $-2\frac{1}{2}$.
12. -4, $4\frac{1}{2}$.
13. $-5/3$, $4\frac{2}{3}$
14. 8, -2.
15. 1, $-1\frac{1}{2}$.
16. 2, $-1\frac{1}{3}$.
17. -6, 1/6.
18. $3\frac{1}{2}$, $3\frac{1}{2}$.
19. 3, 3.
20. a, a.
21. a, b.
22. m, $m/2$.
23. $4/7$, -2.
24. -1, -4.
25. 6, $3\frac{4}{9}$.
26. 9.
27. 9.
28. 9.
29. 9
30. No solution.
31. 10.
32. 9, 25.
33. -8, -9.
34. 2.
35. -1, $\frac{5 \pm \sqrt{21}}{2}$.
36. 4.
37. 5.

Page 108

1. 4.
2. 9.
3. 1.
4. 16.
5. 4.
6. 9.
7. 25.
8. 9.
9. 9/4.
10. 49/4.
11. 25/4.
12. 25/4.

Pages 111–113

1. (*a*) Real and unequal.
3. (*a*) $k = 4$.
5. (*a*) Min. 1; (2, 1).
6. Concave downward.
7. 4 rd. by 4 rd.; $A = 16$ sq. rd.
8. 64 ft.
9. 5 in.
10. 5 and 5.
11. (*a*) Real and unequal.
12. $m = \pm 4$.
13. (*a*) Yes.
14. 4 in. by 5 in.

Pages 117–118

1. (*a*) (1,0), (*b*) (0,0), 4.
3. (*a*) 3.7 + sec.
4. 5216 ft., 64 ft. per sec.
6. $x^2 = -484y$.
8. 0.4 in.
9. Max. height = 256 ft.
11. 6 ft.
12. $x^2 = 500y$.
13. $x^2 = 800y$.
14. $y^2 = \frac{25}{8}x$.
15. $x^2 = -4\frac{1}{2}y$.
16. 3, 3.64, 5.56
17. $y^2 = 24x$.
18. $x^2 = 100y$.
19. $x^2 = -486y$.

Page 120

6. $y^2 = 76.4x$

Pages 130–131

1. (*a*) $x^2 + 5x + 15$, 26.
2. (*a*) Each one is a root.
3. (*a*) $-3 \pm \sqrt{7}$.
4. $x^2 - 20x + 96 = 0$.
6. $f(1) = 4$.
7. $f(0) = 2$.

Pages 135–136

1. $x^3+8x^2+25x-35=0$.
2. $x^3+12x^2+43x-1=0$.
3. 2.406
4. 2.542
5. 1.588
6. 3.682
7. 4.099
8. −1.8004, 4.8922, 0.9082
9. 9.9666, Two roots imag.
10. 4.117, two roots imag.
11. 2.3569, 2,6920, −2.0489
12. 1.6783, 4.1774, −.8557

13. 2.4272, 2.7569, −4.1842
14. 3.8737, −1.7239, −0.1498
15. 0.8019, −0.5549, −2.2469
16. 1.4422
17. 2.7589
18. 4.0207
19. 3.362
20. 3.271
21. 4.130
22. 0.217
23. 0.92
24. 2.37
25. 2.81
26. 2.08
27. 2.36
28. 0.015

Page 141

1. (*a*) 17.
2. (*a*) 2, 5.
3. (*a*) 20.
5. (*a*) 1, 2.
   (*b*) 1, 5.
   (*c*) 5, 2.
5. (*d*) 1, 4.
   (*e*) 2, 3.
   (*f*) 2, 3.

Page 145

1. 19.
2. − 77.
3. 28.
4. 289.
5. 57.
6. $x_1y_2 - x_1y_3$
7. − 9.
8. 0.
9. $x^2 - y^2$.
10. $x = 1, y = 2, z = 3$.
11. $x = 1, y = 3, z = 4$.
12. $x = 2, y = 3, z = 4$.
13. $x = 1, y = 3, z = 2$.
14. $x = 0, y = 4, z = 5$.
15. $x = 1, y = 2, z = 5$.
16. $x = 1, y = 2, z = 1$.
17. $x = 1, y = 5, z = 3$.

Pages 147–148

1. (*a*) 25.
2. (*a*) 19.
3. (*a*) 21.
4. (*a*) $7x - 4y = 3$.
9. $\frac{c}{2}(a + b - d)$.
10. 454 sq. rds.
11. 5.925 acres.
12. $972.

Pages 156–158

1. (*a*) $dy/dx = 2x$.
2. $dy/dx = - .02x$.
3. (*a*) $dy/dx = 2x + 5$.
4. (*a*) 0.
5. (*b*) (4, − 6).
6. 5 and 5.
7. $a/2$ and $a/2$.
8. 5 and 5.
9. $a/2$ and $a/2$.
10. 1.
11. 2, 6.
12. $a/4$, $3a/4$.
13. 4, 6.
14. 1/2.
15. 25 by 25 rods.
16. $20 \times 20 \times 5$ in.
17. 10 by 10 in.
18. 3 in.
19. 12.9
20. Increasing when $t = 2$.
21. 2 by 8 by 8 in.
22. Radius = height.
23. $b/2$, $h/2$.
24. 5 by 10 in.
25. (± 3, 81).
26. (*a*) $0.08x$.
27. End of 5 seconds.
    400 ft.
28. 14″ by 14″ by 28″.

Pages 162–164

1. $3x^2\Delta x$; 30 cu. in.
2. $4\pi r^2\Delta r$, 3π cu. mm.
3. $2ab$ sq. in.
5. $v = 112 - 32t$.
6. 144π sq. in./sec.
7. 108 k.
8. $\Delta v = 3x^2\Delta x = 9.6$ cu. in.
9. $v = 64$ ft./sec. when $t = 2$.
10. $v = 80 - 32t$.
11. (*a*) $t/2 + 2$.
12. 16 ft.; 5 sec.
13. $dK/dt = mv\, dv/dt$.

14. (*a*) $ds/dt = gt$.
15. $a = 6 - 0.6t$.
16. $dF/dt = 75 - 3t^2$. If $t = 1$, $dF/dt = 72$.
17. Increasing when $t = 2, 5, 10$.
18. $v = 88 - 4t$. If $t = 2$, $v = 80$ ft./sec.
19. When $t = 40$ sec.
20. When $t = 1/20$.
21. $dx/dt = 1$ in./min.
22. 1 in./min., 24π sq. in./min.
23. 800 ft./sec.; 10 000 ft.; 50 sec.
25. 24 cu. in. per sec.
26. 4π, 16π, 64π cu. mm./min.
27. $dr/dt = 2/3\pi$ in./sec.

Pages 168–169

1. (*a*) $dy/dx = 3x^2 + 4x$.
2. (*a*) Slope $= dy/dx = 2$.
3. (*a*) 4.
4. 1.
5. 25, 25.
6. -1.
7. 2.
8. (2, 20), (4, 16).
9. Height $= \frac{1}{2}$ width.
11. (*a*) (3, 1).
12. Edge of base $= 2 \times$ height.
13. Ratio is 1.
14. 1.

Pages 171–172

1. (*a*) $10x(1 + x^2)^4$.
2. (*a*) $5x^{3/2}/2$.
3. (*a*) $3x^{1/2}/2$.
4. 0.16π sq. in./min.
5. 4.32 cu. in./min.
6. 3 in. per sec.
7. 60 sq. in. per sec.
8. -3 units per sec.
9. $-3\frac{3}{4}$ ft./sec.
10. 4/5 mi. per min.
11. $64\sqrt{3}$ ft./sec.
12. 48 ft./sec.
13. (2, 4).
14. 4 mi./hr.
15. $-1/40$ cu. in./sec.

Pages 176–178 *

1. (*a*) $y = x^2/2, x^5/5, x^8/8$, etc.
2. (*a*) $y = x^3/3 + x^2/2 + 3x$.
3. (*a*) $x^4/4$.
4. (*a*) x^6.
6. $y = x^2 + 3x$.
7. $y = x^2 + 7x$.
8. $y = x^2 - 4x + 8$.
9. $y = 2x^2 - 8x$.
10. $y = x^2 - 6x + 12$; (3, 3).
11. $s = 16t - t^2$.
12. $y = \frac{1}{2}x^2$.
13. $43\frac{3}{4}$ ft.
15. $s = 5t + 5$.
16. $y = .01x^2 + 8$.

Pages 180–181

2. $10\sqrt{5}$ sec.
3. 1600 ft.
4. 8 sec.; 256 ft.
5. 6400 ft.
6. 943.4 ft.
7. 3974.4 ft.
8. $v = 96 - 32t$.
9. 256 ft.; 336 ft.
10. $v = -10 - 32t$; 296 ft.
11. (*a*) $v = 6t$.
12. $s = -16t^2 + 12t$.
13. $s = s_0 + v_0t - \frac{1}{2}gt^2$. $v = v_0 - gt$.
14. 85.
15. $101\frac{2}{3}$
16. $252\frac{3}{4}$.
17. 160 ft.
18. 10 ft.
19. 5 sec. (*a*) $37\frac{1}{2}$ sec.
20. 8 sec.; 192 ft.

* Add a constant c to each indefinite integral.

Page 183

1. (*a*) $\frac{1}{2}(x^2+2)^2+c$. **3.** (*a*) $\frac{1}{4}(x^2+2)^2+c$.

Pages 186–187

1. 8/3.
2. $7\frac{1}{3}$.
3. 52.
4. 72.
5. $16\frac{1}{2}$.
6. 4/5.
7. 16.
8. 5/2.
9. $39\frac{1}{8}$.
10. $-4\frac{1}{6}$.
11. $279\frac{7}{12}$.
12. $19\frac{1}{2}$.
13. 140.
14. 36.
15. 28.
16. $46\frac{2}{3}$.
17. $86\frac{2}{3}$.
18. $20\frac{1}{4}$.
19. $10\frac{2}{3}$.
20. 56.
21. $19\frac{1}{3}$.
22. 4/3.
23. $\frac{4}{3}(7\sqrt{7}-2\sqrt{2})$.
24. $51\frac{1}{5}$.
25. $85\frac{1}{3}$.
26. 36.
27. $2\frac{2}{3}$.
28. $5\frac{1}{3}$.
29. $\frac{4}{3}(6\sqrt{6}-2\sqrt{2})$.

Pages 195–196

1. Increases from 0° to 90°.
2. Decreases from 0° to 90°.
3. $(\pi/2, 1)$, $(5\pi/2, 1)$.
4. $(0, 1)$, $(2\pi, 1)$.
5. No.

Pages 199–201

12. 56/65.
13. 8.57 ft.; 21.63 ft.
15. 48.36 ft.
17. 145.47 ft.
19. 28.20 ft.
21. 128.1 ft. nearly.

Pages 204–206

1. (*a*) 3.
2. (*a*) 3/5; 5; −3.
3. 5/6.
4. 36° 52′ +.
5. (*a*) 18° 26′ +.
6. (*a*) $5x-y=7$.
7. 34° 41′ 43″.
8. 39° 33′ 36″.
13. 40° 14′ 4″; 57° 31′ 42″; 82° 14′ 14″.
14. 90°; 71° 34′ nearly; 18° 26′ nearly.

Pages 208–209

1. sin 60° = .86603
2. sin 120° = .86603
3. sin 90° = 1.
4. sin 30° = 0.5000
5. sin 45° = .70711
8. 72 ft.
9. $\alpha=45°$.
11. $\frac{4800}{\pi}$ ft.

Pages 212–213

1. $A = 86° 18'$. $B = 41' 42'$.
2. $A = 63° 23'$. $B = 52° 37'$.
3. $A = 69° 28'.7$ $C = 48° 31'.3$
4. $A = 52° 18'$. $B = 71° 42'$.
5. $B = 47° 36'.3$ $C = 62° 23'.7$
9. 142 500 000 mi.
10. 18.58 mi.
11. 49.89×10^{15} mi.
14. $a = 9.817$; $B = 47° 36'$ nearly.

Pages 216–217

6. (*a*) 20.57, (*b*) 12.36
7. 1.8 mi.
8. 5/6 radian.
9. 29.57°.
10. 27.5 sq. in.
11. 100 sq. in.
12. 1° nearly.
13. 0.04 ft.; 4′ 35″
14. 0.104 mi.
15. 401 400 mi.
16. 847 960 000 mi.
17. 484 000 000 mi. nearly.
18. 3146 mi.
19. 20.9 mi.
20. 497.7 ft.

Pages 218–219

1. 12.1
2. 14.74
3. 78.2
4. 87 mi. per sec. nearly.
6. 5.19 mi. east, 3 mi. north.
7. (9.16, 0.88).
8. 12; 53° 8′.
9. (*a*) 5, $\theta = \tan^{-1} 4/3$.
10. $(6\sqrt{3}, 6)$.
11. (6, 0°), (6, 60°).

Pages 222–223

1. Circle, $r = 5$, $c = (0, 0)$.
7. $\rho \cos (\theta^\circ - 22^\circ) = 4$.
8. $\rho \cos (\theta^\circ - 36^\circ) = 7$.

Pages 225–227

1. (*a*) 19.42 ft., 10.32 ft.
2. 1261.7 lb.
3. 378.5 lb.
4. 352.1 lb.
5. 485.4 sq. ft.
6. $R = 118.5$ lb.
7. 1175.6 lb.
8. 582.1 lb.
9. 800 lb.
10. 4.64 lb.
11. 49.98 acres.
12. 12° 22′.
13. $h = 1021.3$; $v = 797.9$
14. 8° 25′ 6″.
15. $21\frac{7}{9}$ lb., $34\frac{2}{9}$ lb.
16. (*a*) 54 lb.
17. 3.8π sq. ft.
18. 17.43 lb.
19. 36 + lb.
20. 12.6.
22. 309.2 lb.

Pages 231–232

1. (*a*) $x + \sqrt{3}y = 16$.
2. (*a*) $p = 5$.
3. $3\frac{4}{5}$.
4. $\dfrac{15\sqrt{53}}{53}$.
5. 31 sq. units.
6. $\dfrac{25\sqrt{34}}{34}$.
8. $5x + 12y = 10 \pm 65$.
9. $\dfrac{13\sqrt{37}}{37}$.
10. $9, \dfrac{9\sqrt{145}}{29}, \dfrac{3\sqrt{10}}{2}$.
11. (*a*) $12x - 5y = 10 \pm 26$.
12. (*a*) $\frac{1}{3}\sqrt{13}$.
13. Parallel lines.
14. $4x + 3y = 35$.

Page 234

1. $x + 7y = -2$.
   $7x - y = 22$.
2. (*a*) $x - y = 2$.
   $7x + 7y = 18$.
3. $17x - 17y = 4$.
   $7x + 17y = 0$.
   $x = 1/6$.
4. $11x + 172y + 180 = 0$.
   $89x + 68y - 340 = 0$.
5. $3x - 12y - 2 = 0$.
   $9x - 4y - 22 = 0$.

Pages 237–238

1. (*a*) A.P.
2. (*a*) 15, 120.
3. (*a*) $l = 35$, $s = 222$.
4. (*a*) 45.
5. (*a*) 144 ft., 304 ft.
6. $d = \dfrac{l - a}{n - 1}$.
7. $l = \dfrac{2s - na}{n}$.
9. 78.
13. 16 ft.
14. 326″.
15. \$50.50
16. \$26⅓, \$25⅙, \$206.
17. 200 ft.
18. \$406.40

Pages 241–243

1. (*a*) 32, 63.
2. 62.
3. 100 000 000 bu.
4. 256.
6. 294, 441, 661½.
8. 59% nearly.
9. (*a*) 1/64.
10. 200 000 000.
11. 250 000 in 3d gen.
12. 1024.
13. (*a*) $2^{33} - 2$ cents.

Pages 246–247

1. \$1408.53
2. \$879.19
3. \$1231.71
4. 5% nearly.
5. 1967.
6. \$72 999.17
7. 18 yrs. nearly.
8. 3.8% nearly.
9. \$15 680 000 000.
10. 4.5%.
11. 2.47%.
12. \$9778.
13. \$5846.79
14. \$2372.54; \$2418.33
15. 25½ years of age.

Pages 249–250

1. \$5588.65
2. \$3121.59
3. \$12 078.54
4. \$4436.30
5. \$134 959.52 if set aside at end of year.
6. \$64 000.
7. \$37 733.67
8. \$3305.71
9. \$954.05
10. \$11 862.71
11. \$7539.

Pages 251–252

1. \$6978.07
2. \$1373.62
3. \$2567.75
4. \$7329.48
5. \$3069.566
6. \$9368.14

Page 254

1. 12.
2. 14, 98.
3. $2\sqrt{6}$.
4. (*a*) 1⅓
5. 4/9.
6. 23⅓ ft.
7. $r = \sqrt[12]{2}$.
8. 8.76

Pages 258–259

1. (*a*) $x^3 + 3x^2y + 3xy^2 + y^3$.
9. \$325.78

Pages 264–266

5. (*a*) 29.92 in.
7. (*a*) 23.93 in.
8. 36 654 ft.
9. 3.5 mi. nearly.
10. 16.47 in.
12. 48 sec.
13. $k = 0.20$
14. $i = 0.14$

Pages 267–268

Answers calculated by means of a three-place table.

1. \$2056.
2. 41 109.
3. 13.36 in.
4. \$54 900.
5. 0.135 gram.
6. 20 100.
7. 1312°.
8. 414 ft.
9. 1484 lb.
10. 2009.
11. 54 600.
12. 33 mg.
13. \$1822; \$1790.84
14. 26%.
15. 9 hrs. approx.
16. 294.4
17. 4416°.

Pages 271–273

2. (*a*) 7. 3. (*a*) 10. 4. (*a*) 55.3 +. 7. 10.9 ft.

Pages 276–278

1. $y = 2x \pm 5\sqrt{5}$.
2. (*a*) $3x + 4y = 50$.
3. (*a*) $3x + 4y = 43$.
4. (*a*) $y = 2x \pm 4\sqrt{5}$.
5. $y = 2x \pm 15$.
6. $y = 3x - 3 \pm 4\sqrt{10}$.
7. Yes.
8. (*a*) $y = x \pm 4\sqrt{2}$.
9. 81.4 +.
10. Slope = $-4/3$.
11. Slope = -2.
12. Slope = $-3/4$.
13. $r = 113$ ft.

Pages 280–281

1. Vertex at (0, 0); focus at (1, 0).
9. (*a*) $p = 3$.
10. $x^2 = -360y$.
11. $x^2 = -100y$.
12. $x^2 = 320y$; 15, 15.31, 16.25 ft.
13. $(y - 3)^2 = 12(x - 5)$.
14. 24, 24⅛, 25⅞ ft.
15. $x^2 = 4025.3y$; 2.48 ft.
16. $y^2 = 72x$, 18 in.
17. 2 ft.

Pages 283–284

1. $x - 3y + 9 = 0$.
2. $x + 2y = 2$.
3. $x - 2y + 8 = 0$.
5. $x - y + 5 = 0$.
7. $4x - 2y + 1 = 0$.
9. $50x - 10y + 3 = 0$.
11. $y = x + 4$.
13. $2x + 6y + 27 = 0$.
15. $x - 2y + 4 = 0$.
20. 19⅛ ft., 17⅞ ft., etc.
21. $x^2 = -180y$.

Pages 289–291

4. (*a*) $x^2/36 + y^2/25 = 1$.
6. $e = \sqrt{3}/2$.
7. (*a*) (6, 0), (0, 3).
8. (*a*) $x^2/64 + y^2/25 = 1$.
9. (*a*) $x^2/36 + y^2/20 = 1$.
10. $x^2/400 + y^2/300 = 1$.
11. $a = 13$, $b = 12$.
12. 20, 19.6, 18.3, 16 ft.
13. 12 ft.
15. $x^2/(76)^2 + y^2/(37.8)^2 = 1$.
16. 4, 4.34, 5.38 ft.
17. $x^2/(93\,000\,000)^2 + y^2/(92\,980\,000)^2 = 1$.
19. $x^2/25 + y^2/16 = 1$.
20. An ellipse.

Pages 292–293

1. $x^2/324 + y^2/81 = 1$.
2. (*a*) 28π; (*b*) 11π.
3. $5\frac{1}{4}\pi$ sq. in.
4. $\frac{1125\pi}{4}$ ft.
6. 432π cu. in.
7. $x^2/48 + y^2/36 = 1$.
8. (*a*) $a = 20$, $b = 12$.
9. 0.587 sec.
10. $\frac{(x-5)^2}{25} + \frac{(y-4)^2}{16} = 1$.
11. $b = 91\,600\,000$ miles.

Pages 296–298

1. (*a*) $y = 2x \pm \sqrt{73}$.
2. Eq. of tangent is $x + 2y = 4$.
   Eq. of normal is $2x - y = 3$.
3. x-intercept $= 8\frac{1}{3}$.
   y-intercept $= 2\frac{1}{2}$.
4. (*a*) $6\frac{1}{4}$. (*b*) $6\frac{2}{3}$.
7. $27x + y = 82$.
8. (*a*) ± 4, ± 3.
9. $x - \sqrt{3}y \pm \sqrt{31} = 0$.
10. $-.45$
11. 53° 8′ nearly.
12. Eq. of tangent is $3x + 5\sqrt{3}y = 60$.

Pages 302–303

3. (*a*) $x^2/36 - y^2/9 = 1$.
5. (*a*) $x^2/16 - y^2/9 = 1$.
7. $\pm 15\sqrt{5}/\sqrt{34}$, $\pm 3\sqrt{11}/\sqrt{34}$.
8. $x^2/16 - y^2/9 = 1$.
9. $x^2/576 - y^2/100 = 1$.

Pages 305–306

1. $x^2/9 - y^2/16 = 1$.
2. $x^2/144 - y^2/25 = 1$.
4. $e = \sqrt{2}$.
5. 90°.
7. $5x^2 - 4y^2 + 24y - 56 = 0$.
8. (*a*) $y = \pm \frac{3}{5}x$.
13. $pv = 920$.
16. (*a*) $x = 3$, $y = 4$.

Pages 309–310

1. (*a*) $5x - 6y = 19$.
2. (*a*) $y = 2x \pm 4\sqrt{2}$.
3. $xx_1 + yy_1 = 25$.
4. $x \pm 3y = 20$.
5. $y = x \pm 5\sqrt{2}$.
6. $3x - \sqrt{3}y + 1 = 0$.
7. $y = \sqrt{3}x \pm \sqrt{59}$.
8. $4x_1/9y_1$.
9. $\frac{25}{41}\sqrt{41}$.

Page 313

7. $(x-5)^2 + (y-2)^2 = 49$.
8. $(y-5)^2 = 12(x-3)$.
9. $(x-3)^2/49 + (y+4)^2/25 = 1$.
10. $(x-6)^2/36 + (y-4)^2/16 = 1$.
11. $(x-5)^2/16 - (y-6)^2/9 = 1$.
13. $(x-4)^2/25 + (y-4)^2/16 = 1$.
14. $(x-3)^2/25 + y^2/11 = 1$.
15. $5x - y - 7 \pm 5\sqrt{26} = 0$.

Pages 317–318

1. (*a*) $\sqrt{110}$.
2. (3, 4, $6\frac{1}{2}$).
3. (*a*) ($3\frac{1}{2}$, 5, $6\frac{1}{2}$).
4. (4, 5, 4).
5. (5, 10, 8).
6. 1 to 2.
9. 4, 8, 3.
10. (*a*) $2\sqrt{14}$.
11. (*a*) 7.
12. 5.5; $\sqrt{217}/2$, $\sqrt{118}/2$.

Pages 319–320

1. 4, 6, 7.
2. 6, 3, 2.
4. $\cos\alpha = 6/7$, $\cos\beta = 2/7$, $\cos\gamma = 3/7$.
5. $\alpha = \beta = \gamma = \cos^{-1} 1/\sqrt{3}$.
6. $\cos\alpha = 1/\sqrt{10}$, $\cos\beta = 3/\sqrt{10}$, $\cos\gamma = 0$.
7. 45°.
8. Four.
9. 60°.
10. 60°.
11. Parallel to yz plane.
14. $\cos\alpha = \frac{8}{10}\sqrt{2}$, $\cos\beta = \frac{2}{5}\sqrt{2}$, $\cos\gamma = \frac{1}{2}\sqrt{2}$.
15. (*a*) $l = \sqrt{69}$.
16. $\cos\gamma = \sqrt{11}/4$.

Pages 321–322

2. $\cos^{-1} 13/21$.

Pages 324–326

1. (*a*) $x = 5$, $x = -4$.
2. (*a*) $z = 10$, $z = -2$.
3. $x = 0$, $y = 0$, $z = 0$.
4. $z = K$, $y = K$, $x = K$.
5. $x = 0{,}5$; $y = 0{,}3$; $z = 0{,}8$.
6. $x^2 + y^2 + z^2 = 49$.
7. $(x-2)^2 + (y+3)^2 + (z-5)^2 = 100$.
8. $(x-4)^2 + (y-7)^2 + (z-6)^2 = 14$.
9. $x^2 + y^2 + z^2 - 6x = 16$.
12. (*a*) (2, 3, 1), $r = 6$.
13. $c = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $r = 1\frac{1}{2}$.
14. (*a*) $y = 0$, $z = 0$.
15. (*a*) $x^2 + y^2 = 9$.

Page 328

1. (*a*) $x^2 + y^2 = 25$.
2. (*a*) $y^2 = 6x$.
4. (*a*) $x^2 + y^2 + z^2 = 25$.
5. (*a*) $y^2 + z^2 = 9$.

Pages 331–332

1. $41\frac{2}{3}\pi$.
2. 36π.
3. $\frac{4}{3}\pi$.
4. $69\frac{1}{3}\pi$.
5. 126π.
6. 8π.
7. 32π.
8. $161\frac{2}{3}\pi$.
9. 200π.
10. 25π.
11. $4\pi a x^2_1$.
12. $1738\frac{2}{3}\pi$.
13. $\frac{4}{3}\pi r^3$.
14. $\frac{4}{3}\pi a^3$.
15. $129\frac{3}{5}\pi$.
16. $48\frac{3}{5}\pi$.
17. $\frac{4}{3}\pi ab^2$.
18. $\frac{4}{3}\pi a^2 b$.
19. $\frac{4}{3}\pi b^2 a$.
20. $(32\pi a^3)/105$.
21. $2340\frac{4}{7}\pi$.

Pages 334–335

1. 1440 foot-pounds.
2. 56 520 foot-pounds.
3. 1.1 h.p.
4. No, only 48/55 h.p.
5. 28 h.p.
6. 59 h.p. nearly.
7. $2\frac{2}{3}$ h.p.
8. 55.7 + h.p.
9. 1200 h.p.
10. 45 in. lb.
11. $158\frac{1}{3}$ ft. lb.
12. 60 ft. lb.
13. (*a*) 1/3600 lb.
14. 98 in. lb.
15. 14 $166\frac{2}{3}$ ft. lb.

Pages 336–337

1. 12.
2. 20.
3. 16.
4. Yes.
5. 20.
6. 150.
7. 2 ways.
8. 6 ways.
9. 36 ways.
10. 12 ways
11. 45.

Pages 339–340

1. 4, 12.
2. 72.
3. 64.
4. 30.
5. 650.
6. 6.
7. 1956.
8. 120.
9. 28.
10. 325.
11. 40 symbols.

Pages 341–342

1. 220.
2. 45.
3. 28.
4. 435, 4060.
5. 20^6.
6. 280.
7. 9!
8. 15.
9. 70.
10. 63.
11. 31.
12. 45.
13. 15.
14. 1690.

Page 343

1. 1/2.
2. (*a*) 1/4.
3. (*a*) 1/8.
4. 1/6.
5. (*a*) 1/36.
6. (*a*) 1/216.
7. 0.92
8. 0.85
9. 1/7.
10. No.

Pages 345–346

1. 1/6.
2. 0.93
3. 1/2.
4. 0.6 +
5. 5/16.
6. 0.88
7. (*a*) 0.81 +
8. (*a*) 0.95 +
9. 0.92 −
10. 0.92 −
11. $7121.66
12. $0.50
13. $40.32
14. $6.92
15. $2.
16. $7.49
17. 0.88
18. 0.045
19. $18.75
20. $5665.72

Page 348

1. 1/60.
2. 0.77 −
3. 0.12
4. 0.8

Pages 352–353

5. 4π.
6. $0.01\pi r^2$.
8. $2\pi hre$.
9. 1.44 cu. ft.
12. 0.43 of one per cent.
13. 1328.79 ft.
14. 1282.88 ft.
15. 4.05 sq. ft.
17. $200\,000\pi$ mi.

Pages 356–357

1. (*a*) Arithmetic mean.
2. 81.6%.
3. 81.75%.
4. 11.6%.
6. 4.8%.
7. $21.77
8. 28.90%.
9. 18.9%.
10. 90%, 89%.
11. 10.6%.

Pages 362–363

1. $Q_1 = \$695.83$, $m = \$816.44$, $Q_3 = \$952.38$.
2. $Q_1 = 86.33\%$, $m = 96.1\%$, $Q_3 = 105.9\%$.
4. A's. $Q_1 = 6.1$, $m = 7.13$, $Q_3 = 8.1$ B's. $Q_1 = 5.94$, $m = 6.87$, $Q_3 = 7.75$
5. (*a*) 8.38. (*b*) $Q_1 = 7.15$, $m = 8.41$, $Q_3 = 9.65$

Pages 366–367

1. $Q_1 = 79.1$, $m = 85.03$, $Q_3 = 90.21$, Quar. Dev. = 5.55.
2. $Q_1 = 158.75$, $m = 168.4$, $Q_3 = 176.4$, Arith. mean = 167.
3. Men. $Q_1 = 7.89$, $m = 10.12$, $Q_3 = 13$, arith. ave. = 10.4.
   Women. $Q_1 = 9.43$, $m = 11.65$, $Q_3 = 13.97$, arith. ave. = 11.7.

Page 374

1. (*a*) 1/64, (*b*) 3/32, (*c*) 15/64.
5. $(1/2)^{100}$.
6. 1/36, 10/36, 25/36.
7. $1/6^3$, $15/6^3$, $125/6^3$.
9. 50.412
10. $m = 20.312$

Pages 379–380

5. (*a*) $5/2(\sqrt{3} + i)$.
6. (*a*) $x = 2$, $y = 3$.
8. (*a*) $2 \pm 2\sqrt{-1}$.

Pages 381–382

1. $4 + 5i$.
2. $7 + 6i$.
3. $7 + 8i$.
4. $5 + 6i$.
5. $1 + 5i$.
6. $2 + 6i$.
7. $6 + 4i$
8. $10 + 5i$.
9. $2 - 4i$.
11. $24 + 10i$.
12. $12 + 5i$.

Pages 385–386

1. $12i$.
2. $2 + 9i$.
3. $12 + 6i$.
4. $-4 + 3i$.
5. $8 + 16i$.
6. $3 - \sqrt{3} + (3\sqrt{3} + 1)i$.
9. $\pm 1, \pm i$.
10. $2, -1 \pm \sqrt{-3}$.
12. $2, -1 \pm \sqrt{-3}, -2, 1 \pm \sqrt{-3}$.
13. (*a*) $-3i$, $3/2(\pm\sqrt{3} + i)$.
14. $1 - 3i$, 5.

9/20/32 — 9+ 2

9/[illegible]/32 — 6+ 10

9/8/32 — 5+ 6

9/12/32 — 4 2

[illegible] — 7 8-

9/19/32 — 2 — 0

$
8+
7 —

5.7

Math ## 5

Sept. 3

Room 708

Sat

9:30 - 12:30

95
65
55
3)215
71 2/3